NOTES ON BOGOMOLOV STABILITY AND RELATED TOPICS

The purpose of these notes is to collect information on Bogomolov stability and related topics, for use in the curves seminar.

1. Stability and the Bogomolov-Gieseker inequality

1.1. Let $E$ be a rank 2 vector bundle on a smooth surface $X$, and set

$$\Delta(E) := c_1(E)^2 - 4c_2(E).$$

The number $\Delta(E)$ is called the discriminant of $E$. If $\alpha$ and $\beta$ are the Chern roots of $E$ (so that $c_1(E) = \alpha + \beta$ and $c_2(E) = \alpha \beta$) then $\Delta(E) = (\alpha - \beta)^2$. From this, or directly from the defining formula, we see that $\Delta(E) = \Delta(E^*)$ and that $\Delta(E) = \Delta(E \otimes L)$ for any line bundle $L$.

1.2. Theorem (Bogomolov). — $\Delta(E) > 0$ if and only if there exists an exact sequence

$$(1.2.1) \quad 0 \rightarrow S \rightarrow E \rightarrow Q \otimes I_Z \rightarrow 0$$

with $S$ and $Q$ line bundles, $Z$ a 0-dimensional subscheme of $X$ with sheaf of ideals $I_Z$ such that

(i) $(S - Q)^2 > 4 \deg Z$, and

(ii) $(S - Q) \cdot H > 0$ for any ample divisor $H$.

1.3. Remarks. (a) The set

$$(1.3.1) \quad \{ \eta \in \text{NS}(X)_\mathbb{R} \mid \eta^2 > 0 \text{ and } \eta \cdot H > 0 \text{ for all ample } H \}$$

is a cone in $\text{NS}(X)_\mathbb{R}$, called the positive cone, which is an open subset of the big cone. The condition that $\eta^2 > 0$ implies that either $\eta$ or $-\eta$ is a big class and therefore that either $\eta$ or $-\eta$ is effective. Given any ample class $H$, this means that either $\eta \cdot H > 0$, so that $\eta$ is effective (and big) or $\eta \cdot H < 0$ so that $-\eta$ is effective (and big). In particular, given that $\eta^2 > 0$ the condition that "$\eta \cdot H > 0$ for all ample $H$" in (1.3.1) is equivalent to "$\eta \cdot H > 0$ for some ample $H$".

An equivalent description of the positive cone is that the condition $\eta^2 > 0$ defines a locus in $\text{NS}(X)_\mathbb{R}$ with two connected components (each a cone), and the positive cone is the connected component which contains the classes of ample line bundles.

Conditions (i) and (ii) of Theorem 1.2 above say that $(S - Q)$ is in the positive cone, and in particular that $(S - Q)$ is big.

Note that the cone defined by (1.3.1) is usually not all of the big cone – there can be big classes $\eta$ with $\eta^2 < 0$. As a simple example, let $\pi: X \rightarrow \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at a single point with exceptional divisor $E$. Then for any $n \geq 0$ the class $\eta := \pi^* \mathcal{O}_{\mathbb{P}^2}(1) + nE$
is big (since it is a big class plus an effective class) but \( \eta^2 = 1 - n^2 \), which is negative as soon as \( n \geq 2 \).

(b) Given the sequence (1.2.1) we have \( c_1(E) = S + Q \) and \( c_2(E) = S \cdot Q + \deg Z \), so that \( \Delta(E) = (S - Q)^2 - 4 \deg Z \). Thus, given that a decomposition such as (1.2.1) exists then condition (i) is the same as the condition \( \Delta(E) > 0 \). By remark (a) above, once (i) holds, then either (ii) holds for one (and hence for all) ample classes \( H \), or else \( (S - Q) \cdot H < 0 \) for one (and hence for all) ample classes \( H \).

(c) A rank 2 vector bundle \( E \) can only have one decomposition as in (1.2.1) satisfying conditions (i) and (ii) above, i.e., the bundles \( S \) and \( Q \), and the inclusion \( S \hookrightarrow E \) are unique (and thus \( Z \) is also determined). To see this suppose we had another such decomposition with subbundle \( S' \) and quotient \( Q' \). By composition we would obtain a nonzero morphism \( S' \rightarrow Q \), so that \( Q = S'(D) \) for some effective divisor \( D \). We will write this relation in additive form as \( S' = Q - D \). From \( S + Q = c_1(E) = S' + Q' \) we then conclude that \( Q' = S + D \) for the same divisor \( D \). But then
\[
(S' - Q') = ((Q - D) - (S + D)) = Q - S - 2D = -((S - Q) + 2D),
\]
and so \( S' - Q' \), being the negative of an effective class, cannot itself be effective, i.e., \( S' - Q' \) fails condition (ii).

In the words of Miles Reid (in [R] below):

“\text{This unicity [i.e., uniqueness of decomposition], together with the fact that we have a purely numerical criterion for instability is in striking contrast with the case of vector bundles over curves.}”

Possible sources for this theorem are

[R] M. Reid, Bogomolov’s theorem \( c_1^2 \leq 4c_2 \), Proceedings of the International Symposium on Algebraic Geometry.

[L] R. Lazarsfeld (with the assistance of Guillermo Fernández del Busto), Lectures on linear series, IAS/Park.

1.4. Theorem (Bogomolov). — If \( H \) is any polarization (i.e, ample line bundle) and if \( E \) is \( H \)-semistable then \( \Delta(E) \leq 0 \).

This theorem has an extension to bundles of arbitrary rank \( r \geq 2 \) on smooth surfaces:

1.5. Theorem (Bogomolov-Miyaoka?). — Let \( X \) be a smooth projective surface and \( E \) a vector bundle of rank \( r \geq 2 \) on \( X \). If \( E \) is \( H \)-stable with respect to some ample line bundle \( H \) then
\[
c_2(E) \geq \frac{r - 1}{2r} c_1(E)^2.
\]

A proof of this result can be found in Lecture III (Theorem 3.4, page 69) of


1.6. Theorem (Miyaoka, [M1, Lemma 8]). — Again assume that \( X \) is a smooth projective surface, and \( E \) a rank 2 vector bundle. If there is a (not necessarily effective) divisor \( D \) on \( X \) such that the linear series \( |O_E(1) - \pi^*D| \) contains an irreducible divisor (where \( P(E) \)
is the projectivization parameterizing one-dimensional quotients and \( \pi : P(E) \to X \) the morphism to \( X \), then
\[ D \cdot c_1(E) \leq c_2(E) + D^2. \]

1.7. **Theorem (Miyaoka, [M1, Theorem 3]).** — Let \( X \) be a smooth projective surface, and \( \mathcal{F} \subset \Omega_X \) a locally free sheaf of rank 2 such that \( \text{det}(\mathcal{F}) \) is nef. If \( \text{Sym}^m(\mathcal{F})(-D) \) has a nontrivial global section for \( m > 0 \) then \( D \cdot \text{det}(\mathcal{F}) \leq \min(mc_2(\mathcal{F}), 0) \). (Here again \( D \) does not have to be effective.)


2. **Applications**

2.1. **Theorem (Bogomolov-Miyaoka-Yao inequality).** — Let \( X \) be a surface of general type. Then \( c_1(X)^2 \leq 3c_2(X) \). By Yao’s theorem (and the Hirzebruch proportionality principle) we have equality if and only if \( X \) is a ball quotient.

This is Theorem 4 of [M1].

An important application of Bogomolov instability in the theory of surfaces is given in


As a sample result (among the many in the paper), we have

2.2. **Theorem (Reider).** — Let \( X \) be a smooth surface and \( L \) a nef line bundle on \( X \). If \( p \) is a base point of \( |K_X + L| \) then there exists an effective curve \( C \) passing through \( p \) such that either (a) \( L \cdot C = 0, C^2 = -1 \), or (b) \( L \cdot C = 1, C^2 = 0 \).

2.3. **Corollary (Fujita conjecture in dimension 2).** — Let \( X \) be smooth surface and \( A \) an ample line bundle. Then \( |K_X + 3A| \) is globally generated and \( |K_X + 4A| \) is very ample.

A completely different use of the ideas of Bogomolov stability is given by

2.4. **Theorem (Paoletti).** — Let \( X \) be a smooth surface, \( C \subset X \) a reduced irreducible curve (but not necessarily smooth) with \( C^2 > 0 \). Let \( \varphi : C \to \mathbb{P}^1 \) be a morphism. If \( \deg(\varphi) < \sqrt{C^2} - 1 \) then there exists a morphism \( \psi : X \to \mathbb{P}^1 \) extending \( \varphi \). Furthermore the restriction map
\[ H^0(X, \psi^*\mathcal{O}_{\mathbb{P}^1}(1)) \to H^0(C, \varphi^*\mathcal{O}_{\mathbb{P}^1}(1)) \]
is injective.

Paoletti’s theorem is actually more general — the full statement applies to smooth projective variety \( X \) (of \( \dim \geq 2 \)), a reduced irreducible divisor \( Y \subset X \) (required to be ample if \( n \geq 3 \)) and gives a similar criterion for a morphism from \( Y \) to \( \mathbb{P}^1 \) to extend to a morphism from \( X \) to \( \mathbb{P}^1 \). The full statement is in

A very nice application of Paoletti’s result is the computation of the gonality of curves on Hirzebruch surfaces:

2.5. Theorem (Martens). — Let $C$ be a reduced and irreducible curve on a Hirzebruch surface $X = F_e$ and assume that $C$ is not a fibre. Then the minimal degree $k$ of a morphism of $C$ onto $\mathbb{P}^1$ is computed by a ruling of $X$ unless $C \sim d(B + F)$ on $X = F_1$ (i.e, $e = 1$) with $d \geq 2$, in which case $C$ is isomorphic to a plane curve of degree $d$ and $k = d - 1$.

This is the main theorem of


3. RELATED MATERIAL

3.1. Theorem (Bogomolov). — Let $X$ be a smooth projective variety, and $L \subset \Omega_X$ a line bundle. Then $\kappa(L) \leq 1$ (i.e, the Kodaira dimension of $L$ can be at most one).

This result is needed in the proof of Theorem 2.1, the “BMY bound”.

3.2. Theorem (BDPP, CDP). — Let $X$ be a projective variety. Then $X$ is rationally connected if and only if for every line bundle $L \subset \Omega_X^p$ (and all $1 \leq p \leq \dim(X)$, $L$ is not pseudo-effective. Equivalently, $X$ is not rationally connected if and only if there is some $p$ and pseudo-effective line bundle $L \subset \Omega_X^p$.

The papers of interest for this result are:


3.3. Theorem (Mehta-Ramanathan). — Let $X$ be a smooth projective variety of dimension $n$, $h = (H_1, \ldots, H_{n-1})$ a sequence of ample divisors, and $E$ a vector bundle on $X$. Then $E$ is semistable on $X$ if and only if $E|_C$ is semistable, where $C$ is a very general complete intersection curve of type $D_1 \cdot D_2 \cdots D_{n-1}$, with $D_i \in |m_i H_i|$, $m_i \gg 0$.

This extremely useful theorem (called the ‘Mehta-Ramanathan restriction theorem’) is proved in

[MR] V.B. Mehta and A. Ramanathan, Semistable sheaves on projective varieties and their restriction to curves.

3.4. Theorem (Miyaoka). — Let $X$ be a normal projective variety. If $X$ is not uniruled then $\Omega_X$ is generically nef.

Will also need to define “generically nef” and related ideas. Relevant papers are


[S-B] N. Shepherd-Barron, Miyaoka’s theorem on the seminegativity of $T_X$, in Flips and abundance for algebraic threefolds; ed. J. Kollár; Astérisque 211 (1992), 103–114
Finally, in relation to Theorem 3.1, one can look at subsheaves $\mathcal{F} \subset \Omega^p_X$ and study the Kodaira dimension of $\det(\mathcal{F})$. 