1. Compute each of the following vectors.

(a) \[
\begin{bmatrix}
5 & 1 \\
2 & 1 \\
7 & 3
\end{bmatrix}
\begin{bmatrix}
2 \\
-3
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
8 & -7 & -4 \\
1 & 5 & 3
\end{bmatrix}
\begin{bmatrix}
3 \\
2
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
-1 & 3 \\
4 & 0
\end{bmatrix}
\begin{bmatrix}
6 \\
2
\end{bmatrix}
\]

(d) \[
\begin{bmatrix}
2 & 3 & 1 \\
0 & 1 & -4 \\
2 & 5 & 7
\end{bmatrix}
\begin{bmatrix}
-3 \\
5 \\
2
\end{bmatrix}
\]

Solution.

(a) \[
\begin{bmatrix}
5 & 1 \\
2 & 1 \\
7 & 3
\end{bmatrix}
\begin{bmatrix}
2 \\
-3
\end{bmatrix} = 2 \begin{bmatrix}
5 \\
2 \\
7
\end{bmatrix} - 3 \begin{bmatrix}
1 \\
1 \\
3
\end{bmatrix} = \begin{bmatrix}
7 \\
1 \\
5
\end{bmatrix}.
\]

(b) \[
\begin{bmatrix}
8 & -7 & -4 \\
1 & 5 & 3
\end{bmatrix}
\begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix} = 3 \begin{bmatrix}
8 \\
1
\end{bmatrix} + 2 \begin{bmatrix}
-7 \\
5
\end{bmatrix} + 1 \begin{bmatrix}
-4 \\
3
\end{bmatrix} = \begin{bmatrix}
6 \\
16
\end{bmatrix}.
\]

(c) \[
\begin{bmatrix}
-1 & 3 \\
4 & 0
\end{bmatrix}
\begin{bmatrix}
6 \\
2
\end{bmatrix} = 6 \begin{bmatrix}
-1 \\
4
\end{bmatrix} + 2 \begin{bmatrix}
3 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
24
\end{bmatrix}.
\]

(d) \[
\begin{bmatrix}
2 & 3 & 1 \\
0 & 1 & -4 \\
2 & 5 & 7
\end{bmatrix}
\begin{bmatrix}
-3 \\
5 \\
2
\end{bmatrix} = -3 \begin{bmatrix}
2 \\
0 \\
2
\end{bmatrix} + 5 \begin{bmatrix}
3 \\
1 \\
5
\end{bmatrix} + 2 \begin{bmatrix}
-4 \\
-1 \\
7
\end{bmatrix} = \begin{bmatrix}
11 \\
-3 \\
33
\end{bmatrix}.
\]

2. Compute the standard matrix for each of the following linear transformations.

(a) The linear transformation \(T: \mathbb{R}^3 \rightarrow \mathbb{R}\) given by \(T(\vec{v}) = \vec{u} \cdot \vec{v}\), where \(\vec{u} = (2, 4, 3)\).

(b) The linear transformation \(T: \mathbb{R}^3 \rightarrow \mathbb{R}^3\) from given by \(T(\vec{v}) = \vec{u} \times \vec{v}\) where \(\vec{u} = (u_1, u_2, u_3)\). (This is the linear transformation that appeared in H5, Question 3.)

(c) The linear transformation \(T: \mathbb{R}^3 \rightarrow \mathbb{R}^3\) which is “rotate by \(\pi/2\) counterclockwise around the \(z\)-axis”. (Here “counterclockwise” means if you are on the positive \(z\)-axis looking down at the \(xy\)-plane, you want to rotate it counterclockwise.)
Solution. If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear transformation, then the standard matrix for \( T \) is the \( m \times n \) matrix whose columns are \( T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n) \).

(a) Since \( T(\vec{e}_1) = 2, T(\vec{e}_2) = 4, \) and \( T(\vec{e}_3) = 3 \) the standard matrix for \( T \) is \[
\begin{bmatrix}
2 & 4 & 3
\end{bmatrix}.
\]

(b) Since \( T(\vec{e}_1) = \vec{u} \times \vec{e}_1 = (0, u_3, -u_2), T(\vec{e}_2) = \vec{u} \times \vec{e}_2 = (-u_3, 0, u_1), \) and \( T(\vec{e}_3) = (u_2, -u_1, 0) \) the standard matrix for \( T \) is
\[
\begin{bmatrix}
0 & -u_3 & u_2 \\
u_3 & 0 & -u_1 \\
-u_2 & u_1 & 0
\end{bmatrix}.
\]

(c) Since \( T(\vec{e}_1) = \vec{e}_2, T(\vec{e}_2) = -\vec{e}_1, \) and \( T(\vec{e}_3) = \vec{e}_3 \) (see the picture below), the standard matrix for \( T \) is
\[
\begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

3. Given a line \( L \) in \( \mathbb{R}^2 \), projection onto \( L \) is the function \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) which sends every point in \( \mathbb{R}^2 \) to the nearest point on \( L \), as shown in the diagram at right.

For any \( m \), let \( T_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the projection onto the line with slope \( m \) through the origin. This turns out to be a linear transformation, something you can assume when doing the question.

(a) Find the standard matrix for \( T_m \), and explain your steps.

(b) As \( m \rightarrow \infty \), what happens to the line of slope \( m \)? What happens to the matrix associated to \( T_m \)? Does this make sense?

**Suggestion for (a):** Start by finding two vectors where it is easy to understand the result of applying \( T_m \), and then use linear combinations to deduce what \( T_m \) does to \( \vec{e}_1 \) and \( \vec{e}_2 \). (Another possibility : Use the projection formula.)
Solution.

(a) The standard matrix for $T_m$ is $A_m = \begin{bmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{1}{1+m^2} \end{bmatrix}$.

Following the suggestions, here are two ways to solve this problem.

**Solution I:** There are two vectors in $\mathbb{R}^2$ where it is clear what the projection has to do: for vectors on the line of slope $m$ the projection $T_m$ does nothing, and for vectors perpendicular to the line, the projection $T_m$ sends them to $\vec{0} = (0, 0)$.

A vector lying on the line of slope $m$ is $\vec{v}_1 = (1, m)$, and a vector perpendicular to this line is $\vec{v}_2 = (-m, 1)$. Writing out the previous observations in mathematical notation, we know that

$$T_m \left(\begin{bmatrix} 1 \\ m \end{bmatrix}\right) = \begin{bmatrix} 1 \\ m \end{bmatrix} \quad \text{and} \quad T_m \left(\begin{bmatrix} -m \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

In order to see what $T_m$ does to $\vec{e}_1$ and $\vec{e}_2$ we can use the same strategy as in **H5, Question 1**: write $\vec{e}_1$ and $\vec{e}_2$ as linear combinations of $\vec{v}_1$ and $\vec{v}_2$, and use the linearity of $T_m$ to figure out what happens to $\vec{e}_1$ and $\vec{e}_2$.

We can write $\vec{e}_1$ and $\vec{e}_2$ as

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{1+m^2} \begin{bmatrix} 1 \\ m \end{bmatrix} - \frac{m}{1+m^2} \begin{bmatrix} -m \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{m}{1+m^2} \begin{bmatrix} 1 \\ m \end{bmatrix} + \frac{1}{1+m^2} \begin{bmatrix} -m \\ 1 \end{bmatrix}$$

which is something we can work out either by inspection, or in the usual way, by row reducing the matrix describing this linear algebra problem

$$\begin{bmatrix} 1 & -m & 1 & 0 \\ m & 1 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ 0 & 1 & -\frac{m}{1+m^2} & \frac{1}{1+m^2} \end{bmatrix}.$$ 

Using the linearity of $T$, we can now conclude that

$$T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \frac{1}{1+m^2} T \left(\begin{bmatrix} 1 \\ m \end{bmatrix}\right) + \frac{m}{1+m^2} T \left(\begin{bmatrix} m \\ -1 \end{bmatrix}\right)$$

$$= \frac{1}{1+m^2} \begin{bmatrix} 1 \\ m \end{bmatrix} + \frac{m}{1+m^2} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{1+m^2} \\ \frac{m}{1+m^2} \end{bmatrix}.$$
and

\[
T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \frac{m}{1 + m^2} T \left( \begin{bmatrix} 1 \\ m \end{bmatrix} \right) - \frac{1}{1 + m^2} T \left( \begin{bmatrix} m \\ -1 \end{bmatrix} \right)
\]

\[
= \frac{m}{1 + m^2} \begin{bmatrix} 1 \\ m \end{bmatrix} - \frac{1}{1 + m^2} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{m}{1+m^2} \\ \frac{m^2}{1+m^2} \end{bmatrix} \]

This gives the standard matrix above.

**Solution II:** Use the projection formula. Let \( w = (1, m) \), which is a vector on the line of slope \( m \). The linear transformation \( T_m \) is projection onto \( \vec{w} \), and we know the formula for this: \( \text{proj}_{\vec{w}}(\vec{v}) = \left( \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w} \).

To find the standard matrix of the linear transformation, we just have to see where \( \vec{e}_1 \) and \( \vec{e}_2 \) go. The projection formula tells us that

\[
T_m(\vec{e}_1) = \text{proj}_{\vec{w}}(\vec{e}_1) = \left( \frac{\vec{e}_1 \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w} = \left( \frac{1}{(1 + m^2)} \right) \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} \frac{1}{1+m^2} \\ \frac{m}{1+m^2} \end{bmatrix},
\]

\[
T_m(\vec{e}_2) = \text{proj}_{\vec{w}}(\vec{e}_2) = \left( \frac{\vec{e}_2 \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w} = \left( \frac{m}{(1 + m^2)} \right) \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} \frac{m}{1+m^2} \\ \frac{m^2}{1+m^2} \end{bmatrix}
\]

And that again gives the standard matrix shown above.

(b) As \( m \to \infty \) the line of slope \( m \) becomes vertical. As \( m \to \infty \) the matrix \( A_m \) becomes

\[
\lim_{m \to \infty} A_m = \lim_{m \to \infty} \left[ \begin{bmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{1}{1+m^2} \end{bmatrix} \right] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

which is the matrix for projection onto the \( y \) axis, just as expected.

4. In this problem we will finish the proof of the theorem from the class of Thursday, October 20th.

Let \( A \) be an \( m \times n \) matrix, and define a function \( T : \mathbb{R}^n \to \mathbb{R}^m \) by the rule \( T(\vec{v}) = A\vec{v} \) for each \( \vec{v} \in \mathbb{R}^n \). We want to show that \( T \) is a linear transformation.
In order to prove this, it will help to explicitly write out what “$A\vec{v}$” means. Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ be the column vectors of $A$. Then for $\vec{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $A\vec{v}$ is the vector $x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n$ in $\mathbb{R}^m$.

Therefore, the issue really is: Show that the function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by the rule

$$T(x_1, x_2, \ldots, x_n) = x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n$$

is a linear transformation.

Your mission in this question: Show it!

**Solution.**

**Addition Test:** For vectors $\vec{v} = (x_1, x_2, \ldots, x_n)$ and $\vec{w} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$,

\[
T(\vec{v} + \vec{w}) = T(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = (x_1 + y_1)\vec{w}_1 + (x_2 + y_2)\vec{w}_2 + \cdots + (x_n + y_n)\vec{w}_n \\
= x_1\vec{w}_1 + y_1\vec{w}_1 + x_2\vec{w}_2 + y_2\vec{w}_2 + \cdots + x_n\vec{w}_n + y_n\vec{w}_n \\
= (x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n) + (y_1\vec{w}_1 + y_2\vec{w}_2 + \cdots + y_n\vec{w}_n) \\
= T(x_1, \ldots, x_n) + T(y_1, \ldots, y_n) = T(\vec{v}_1) + T(\vec{v}_2).
\]

Since the vectors $\vec{v}$ and $\vec{w}$ were arbitrary, $T$ passes the general addition test.

**Scalar Multiplication Test:** For a vector $\vec{v} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and scalar $c \in \mathbb{R}$,

\[
T(c\vec{v}) = T(cx_1, cx_2, \ldots, cx_n) = (cx_1)\vec{w}_1 + (cx_2)\vec{w}_2 + \cdots + (cx_n)\vec{w}_n \\
= c(x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n) = cT(x_1, \ldots, x_n) = cT(\vec{v}).
\]

Since $c$ and $\vec{v}$ were arbitrary, $T$ passes the general scalar multiplication test. Since $T$ passes both tests, $T$ is a linear transformation.