1. In this problem we will establish a few more relations between the idea of dimension and the ideas of linear dependence and spanning sets. Some of the parts (like (e)) should make intuitive sense. Proving that such statements are actually true is a sign that our definition of dimension is a good one, and also means that we can use these properties freely in the future when we are thinking about subspaces.

Parts (a)–(d) can be solved by using the definition of dimension and the Key Lemma from the class of Thursday, November 17th. Part (f) will likely require the proposition from Tuesday, November 15th.

Suppose that \( W \subset \mathbb{R}^n \) is a subspace, that \( \dim(W) = d \) (i.e., \( W \) is of dimension \( d \)), and that \( \vec{v}_1, \ldots, \vec{v}_m \) are vectors in \( W \).

(a) Explain why you know that there is a set of \( d \) vectors in \( W \) which span \( W \).

(b) If \( m > d \) prove that \( \vec{v}_1, \ldots, \vec{v}_m \) must be linearly dependent.

(c) Explain why you know that there is a set of \( d \) vectors in \( W \) which are linearly independent.

(d) If \( m < d \) prove that \( \vec{v}_1, \ldots, \vec{v}_m \) cannot span \( W \).

Now suppose that \( V \) is another subspace of \( \mathbb{R}^n \), and that \( V \subseteq W \).

(e) Prove that \( \dim(V) \leq \dim(W) \). (SUGGESTION: take a basis for \( V \) and use the Key Lemma.)

(f) If \( \dim(V) = d \) (i.e., the same dimension as \( W \)) prove that \( V = W \). (SUGGESTION: Try proving this by contradiction. Assume that \( V \neq W \) then start with a basis for \( V \) and add to it a vector in \( W \) but outside of \( V \). Then use part (b) of the Proposition from Tuesday, November 15th, to get a contradiction.)

(g) Find the mistake in the following argument. Let

\[
W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}.
\]

We have seen in class that \((1, 0, -1), (0, 1, -1)\) span \( W \). On the other hand, we know that \( \vec{e}_1, \vec{e}_2, \) and \( \vec{e}_3 \) are linearly independent. The Key Lemma says “(# in spanning set) ≥ (# in linearly independent set)”. Applying this to the spanning set \((1, 0, -1), (0, 1, -1)\), and the linearly independent set \( \vec{e}_1, \vec{e}_2, \vec{e}_3 \) gives \( 2 \geq 3 \), which is clearly false. What went wrong?
Solution. Since we will be applying it several times, it is helpful to state the lemma here.

**Key Dimension Lemma:** Let $W \subseteq \mathbb{R}^n$ be a subspace. Suppose that $\vec{u}_1, \ldots, \vec{u}_s$ are vectors which span $W$, and that $\vec{v}_1, \ldots, \vec{v}_t$ are vectors in $W$ which are linearly independent. Then $s \geq t$.

(a) We know that $\dim(W) = d$. By definition of dimension, this means that $W$ has a basis $\vec{w}_1, \ldots, \vec{w}_d$ with exactly $d$ elements. By definition of basis, this means that $\vec{w}_1, \ldots, \vec{w}_d$ span $W$, and are linearly independent. (This question only asks for the “span” part.)

(b) We will prove the statement by contradiction. That is, we will assume that $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent, and show that this leads to a contradiction with something we already know to be true. We then conclude that our assumption must be wrong, that is, that $\vec{v}_1, \ldots, \vec{v}_m$ are linearly dependent.

So, suppose that $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent. Then we can apply the Key Dimension Lemma to conclude that $d \leq m$. Since we know that $m > d$, this is a contradiction. Therefore $\vec{v}_1, \ldots, \vec{v}_m$ are linearly dependent.

(c) Just like in part (a), since $\dim(W) = d$, we know that $W$ has a basis $\vec{w}_1, \ldots, \vec{w}_d$. By definition of basis, this means that $\vec{w}_1, \ldots, \vec{w}_d$ span $W$ and are linearly independent. (This time it is the “linearly independent” part we care about.)

(d) We again prove the statement by contradiction. Suppose that $\vec{v}_1, \ldots, \vec{v}_m$ span $W$. Then by the Key Dimension Lemma (with the $\vec{w}$ and $\vec{v}$ vectors switching roles, now $\vec{v}_1, \ldots, \vec{v}_m$ span and $\vec{w}_1, \ldots, \vec{w}_d$ are linearly independent) we conclude that $m \geq d$. Since we know that $m < d$ this is a contradiction. Therefore the assumption that $\vec{v}_1, \ldots, \vec{v}_m$ span $W$ is incorrect, and $\vec{v}_1, \ldots, \vec{v}_m$ do not span $W$.

(e) Suppose that $m = \dim(V)$, and let $\vec{v}_1, \ldots, \vec{v}_m$ be a basis for $V$. By definition of basis, this means that $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent. Let $\vec{w}_1, \ldots, \vec{w}_d$ be a basis for $W$ (with $d = \dim(W)$ as before). By definition of basis, this means that $\vec{w}_1, \ldots, \vec{w}_d$ span $W$. Applying the Key Dimension Lemma, we get that $d \geq m$. Since $d = \dim(W)$ and $m = \dim(V)$, this is the inequality $\dim(V) \leq \dim(W)$.

(f) Let $\vec{v}_1, \ldots, \vec{v}_d$ be a basis for $V$, and suppose that $V \neq W$ (we are still assuming that $V \subseteq W$ though). Since $V \neq W$, there is a vector $\vec{w} \in W$ such that $\vec{w} \notin V = \text{Span}(\vec{v}_1, \ldots, \vec{v}_d)$. By part (b) of the proposition from Tuesday, November 15th, the vectors $\vec{v}_1, \ldots, \vec{v}_d$, $\vec{w}$ are linearly independent. We now have $d + 1$ linearly independent vectors in $W$. This contradicts the Key Dimension Lemma, since there are a set of $d$ vectors which span $W$ (namely, any basis for $W$), and so the Key Lemma gives the inequality $d \geq d + 1$, which is false. Therefore, there is no such vector $\vec{w}$ outside of $V$ but in $W$, and so $V = W$. 

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(g) The hypotheses of the Key Dimension Lemma are that we have a subspace \( W \), a set of vectors which span \( W \), and a set of vectors in \( W \) which are linearly independent. The error in the argument given is that \( \vec{e}_1, \vec{e}_2, \text{ and } \vec{e}_3 \) are not in the subspace \( W \), and therefore we cannot apply the lemma. (This is a good warning: we can only apply a theorem, lemma, or proposition when we know that all the hypotheses of the theorem, lemma, or proposition have been met.)

NOTES: (1) If we take \( W = \mathbb{R}^n \) (so that \( d = \text{dim}(W) = \text{dim}(\mathbb{R}^n) = n \)), then 1(b) becomes the statement that “If we have more than \( n \) vectors in \( \mathbb{R}^n \), the vectors must be linearly dependent”, which was a lemma from the class of Friday, November 11th. That is, the lemma from that class is a special case of 1(b), which tells us what the general statement is for any subspace.

(2) As a result of (a) and (b), the number \( d \) is the largest size of any set of linearly independent vectors in \( W \). As a result of (c) and (d) the number \( d \) is the smallest number of vectors in any set which spans \( W \). Put together, this tells us that the following three numbers are equal:

- The number of elements in a basis for \( W \).
- The smallest number of vectors in any spanning set for \( W \).
- The largest size of any set of linearly independent vectors in \( W \).

This common number is \( \text{dim}(W) \).

2. Let \( T: \mathbb{R}^6 \rightarrow \mathbb{R}^4 \) be the linear transformation with standard matrix

\[
A = \begin{bmatrix}
2 & 3 & 14 & -3 & 29 & -6 \\
3 & 3 & 15 & 2 & -3 & 12 \\
1 & 1 & 5 & 1 & -3 & 5 \\
2 & 2 & 10 & 4 & -18 & 16
\end{bmatrix}.
\]

The matrix \( A \) has RREF

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & -2 & 3 \\
0 & 1 & 4 & 0 & 5 & -1 \\
0 & 0 & 0 & 1 & -6 & 3 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

(You don’t have to prove that this is the RREF.)

(a) Write the third column of \( A \) as a linear combination of the first two columns of \( A \).

(b) Write the fifth column of \( A \) as a linear combination of the first, second, and fourth columns of \( A \).

(c) Find a basis for \( \text{Im}(T) \).

(d) What is \( \text{dim}(\text{Im}(T)) \)?

(e) Is \( T \) surjective?
(f) Find a basis for Ker(T).

(g) What is dim(Ker(T))?

(h) Is T injective?

(i) What equality does the rank-nullity theorem claim should be true (for this T)?

(j) Is that equality true in this case?

Solution.

(a) From RREF(A) we can see that

\[(\text{third column of RREF}(A)) = 1(\text{first column of RREF}(A)) + 4(\text{second column of RREF}(A)).\]

As discussed in class, this means that the same relation is true for the columns of A, and indeed \((14, 15, 5, 10) = 1(2, 3, 1, 2) + 4(3, 3, 1, 2)\).

(b) Similarly, in RREF(A) we see that

\[(\text{5th column}) = -2(\text{1st column}) + 5(\text{2nd column}) - 6(\text{4th column}),\]

and with the columns of A we similarly have

\[(29, -3, -3, -18) = -2(2, 3, 1, 2) + 5(3, 3, 1, 2) - 6(-3, 2, 1, 4).\]

(c) The first, second, and fourth columns of RREF(A) are the columns with the leading ones. These columns are clearly linearly independent, and any other column in RREF(A) is a linear combination of those columns. The same is then true of the columns of A. Since the columns of A span Im(T), this means that the first, second, and fourth columns of A are a basis for Im(T). I.e.,

\[(2, 3, 1, 2), (3, 3, 1, 2), \text{ and } (-3, 2, 1, 4)\]

are a basis for Im(T).

(d) Since a basis for Im(T) has three vectors, \(\dim(\text{Im}(T)) = 3\). Equivalently, \(\dim(\text{Im}(T)) = \text{Rank}(A) = 3\).

(e) No, \(\dim(\text{Im}(T)) = 3 < 4 = \dim(\mathbb{R}^4)\), so \(\text{Im}(T)\) is not all of \(\mathbb{R}^4\), and T is not surjective. We can also use the characterization from a previous class: the linear transformation maps to \(\mathbb{R}^4\), and \(\text{Rank}(A) = 3 \neq 4\), so T is not surjective.
(f) Vectors \( \vec{v} \) in the kernel are solutions to \( A\vec{v} = \vec{0} \), and this is a problem of parameterizing all solutions to a linear system of equations, a problem we know how to solve. Let \( x_1, \ldots, x_6 \) be the variables for \( \mathbb{R}^6 \). From RREF(\( A \)) we see that \( x_1, x_2, \) and \( x_4 \) are the dependent variables, and \( x_3, x_5, \) and \( x_6 \) the independent variables. Setting \( x_3 = t_1, x_5 = t_2, \) and \( x_6 = t_3 \), we solve for \( x_1, x_2, \) and \( x_4 \) in terms of these parameters:

\[
\begin{align*}
  x_1 + t_1 - 2t_2 + 3t_3 &= 0 \quad \text{or} \quad x_1 = -t_1 + 2t_2 - 3t_3 \\
  x_2 + 4t_1 + 5t_2 - t_3 &= 0 \quad \text{or} \quad x_2 = -4t_1 - 5t_2 + t_3 \\
  x_4 - 6t_2 + 3t_3 &= 0 \quad \text{or} \quad x_4 = 6t_2 - 3t_3.
\end{align*}
\]

In vector form, this is the parameterization

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6
\end{bmatrix} = t_1 \begin{bmatrix} -1 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -3 \\ 1 \\ 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\]

From the parameterization, the vectors \((-1, -4, 1, 0, 0, 0), (2, -5, 0, 6, 1, 0), (-3, 1, 0, -3, 0, 1)\) span \( \text{Ker}(T) \). From class we also know that they are linearly independent (e.g., by looking in the coordinates corresponding to the independent variables). Therefore these vectors form a basis for \( \text{Ker}(T) \).

(g) Since a basis for \( \text{Ker}(T) \) has three vectors, \( \dim(\text{Ker}(T)) = 3 \).

(h) No, \( T \) is not injective. For instance, all the vectors in \( \text{Ker}(T) \) (and there are infinitely many of them) get sent by \( T \) to the same vector \( \vec{0} \in \mathbb{R}^4 \). Equivalently (from a previous proposition in class), since \( T \) has input in \( \mathbb{R}^6 \), for \( T \) to be injective we would need to have \( \text{Rank}(A) = 6 \). Since \( \text{Rank}(A) = 3 \neq 6 \), we conclude that \( T \) is not injective.

(i) Since \( T \) has input in \( \mathbb{R}^6 \), the Rank-Nullity theorem claims that

\[ \dim(\text{Im}(T)) + \dim(\text{Ker}(T)) = 6. \]

(j) This equality holds in this example. We’ve computed that \( \dim(\text{Im}(T)) = 3 \) (part (d)) and \( \dim(\text{Ker}(T)) = 3 \) (part (g)), and \( 3 + 3 = 6 \).