1. Let $A$ be the matrix

$$A = \begin{bmatrix} 3 & 2 & z \\ x & 1 & 3 \\ 2 & y & 5 \end{bmatrix},$$

where $x$, $y$, and $z$ are variables.

(a) Compute the adjoint matrix of $A$ (this will still involve the variables $x$, $y$, and $z$).

(b) Compute the product of the adjoint matrix and $A$.

(c) Compute $\det(A)$.

(d) Assuming that $\det(A) \neq 0$, write down the inverse of $A$ (this will still be a matrix involving $x$, $y$, and $z$).

(e) To show that this method really gives a “universal formula for the inverse”, plug in the values $(x, y, z) = (3, -1, 1)$ into both $A$ and the inverse matrix from part (d), and multiply them to see that it really gives the inverse for $A$.

(f) Do the same for $(x, y, z) = (1, 1, 1)$

Solutions.

$$\text{adj}(A) = \begin{bmatrix} 5 - 3y & yz - 10 & 6 - z \\ 6 - 5x & 15 - 2z & xz - 9 \\ xy - 2 & 4 - 3y & 3 - 2x \end{bmatrix}.$$
(b) Multiplying, we compute that
\[
\begin{bmatrix}
3 & 2 & z \\
x & 1 & 3 \\
2 & y & 5
\end{bmatrix}
\begin{bmatrix}
5 - 3y & yz - 10 & 6 - z \\
6 - 5x & 15 - 2z & xz - 9 \\
xy - 2 & 4 - 3y & 3 - 2x
\end{bmatrix}
= 
\begin{bmatrix}
xyz - 10x - 9y - 2z + 27 & 0 & 0 \\
0 & xyz - 10x - 9y - 2z + 27 & 0 \\
0 & 0 & xyz - 10x - 9y - 2z + 27
\end{bmatrix}.
\]

(c) Since (for 3 \times 3 matrices), \( A \cdot \text{adj}(A) = \text{det}(A) \cdot I_3 \), from part (b) we can see that \( \text{det}(A) = xyz - 10x - 9y - 2z + 27 \).

(d) If \( \text{det}(A) \neq 0 \) the inverse of \( A \) is:
\[
A^{-1} = \frac{1}{\text{det}(A)} \text{adj}(A) = \frac{1}{xyz - 10x - 9y - 2z + 27}
\begin{bmatrix}
5 - 3y & yz - 10 & 6 - z \\
6 - 5x & 15 - 2z & xz - 9 \\
xy - 2 & 4 - 3y & 3 - 2x
\end{bmatrix}.
\]

(e) If \( (x, y, z) = (3, -1, 1) \) then \( A = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 1 & 3 \\ 2 & -1 & 5 \end{bmatrix} \) and the matrix from part (d) becomes
\[
\begin{bmatrix}
8 & -11 & 5 \\
-9 & 13 & -6 \\
-5 & 7 & -3
\end{bmatrix}
\]
with product
\[
\begin{bmatrix}
3 & 2 & 1 \\
3 & 1 & 3 \\
2 & -1 & 5
\end{bmatrix}
\begin{bmatrix}
8 & -11 & 5 \\
-9 & 13 & -6 \\
-5 & 7 & -3
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]
so that matrix really is the inverse of \( A \).

(f) Similarly, if we plug in \( (x, y, z) = (1, 1, 1) \) we get \( A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 3 \\ 2 & 1 & 5 \end{bmatrix} \), and the matrix from part (d) is
\[
\begin{bmatrix}
\frac{2}{7} & -\frac{9}{7} & \frac{5}{7} \\
\frac{1}{7} & \frac{13}{7} & -\frac{8}{7} \\
-\frac{1}{7} & \frac{1}{7} & \frac{1}{7}
\end{bmatrix}
\]
with product
\[
\begin{bmatrix}
3 & 2 & 1 \\
1 & 1 & 3 \\
2 & 1 & 5
\end{bmatrix}
\begin{bmatrix}
\frac{2}{7} & -\frac{9}{7} & \frac{5}{7} \\
\frac{1}{7} & \frac{13}{7} & -\frac{8}{7} \\
-\frac{1}{7} & \frac{1}{7} & \frac{1}{7}
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]
so again the formula really works – it is a universal formula for computing the inverse of $A$.

2. Suppose that $A$ is an $n \times n$ matrix with integer entries, and that $A$ is invertible. Since $A$ is invertible, we can compute the matrix $A^{-1}$. The computations seem much cleaner (and friendlier) when $A^{-1}$ also has only integer entries. The purpose of this question is to figure out when that happens.

(a) Show that if $\det(A) = \pm 1$ then $A^{-1}$ has only integer entries. (Suggestion: Use the expression for the inverse in terms of the adjoint matrix.)

(b) Conversely suppose that $A^{-1}$ also has integer entries. Using the fact that

\[ \det(A) \det(A^{-1}) = \det(I_n) = 1, \]

explain why we must have $\det(A) = \pm 1$.

(c) Conclude that if $A$ is a square matrix with integer entries then $A^{-1}$ has integer entries if and only if $\det(A) = \pm 1$.

Solution. The basic key to the problem is that for any $n \times n$ matrix with integer entries, its determinant (or any $(n-1) \times (n-1)$ sub-determinant) is also an integer. We can see this by induction, or from the determinant formula

\[ \det(A) = \sum_{\sigma \in S_n} \sgn(\sigma) a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)} \cdots a_{\sigma(n)}; \]

the formula only involves multiplying entries of $A$ and then adding these terms together. If all of the $a_{ij}$ are integers then the formula produces an integer.

(a) If $A$ has only integer entries, then by the remark above each entry of $\text{adj}(A)$ is an integer, since each entry is made by computing a the determinant of an $(n-1) \times (n-1)$ submatrix of $A$. If $\det(A) \neq 0$ we have the formula $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$. If $\det(A) = \pm 1$, we then have $A^{-1} = \pm \text{adj}(A)$, so that each entry of $A^{-1}$ is an integer.

(b) Conversely, suppose $A$ is invertible and that both $A$ and $A^{-1}$ have integer entries. By the first remark both $\det(A)$ and $\det(A^{-1})$ are integers. The formula

\[ \det(A) \det(A^{-1}) = \det(I_n) = 1 \]

shows us that $\det(A^{-1}) = 1/\det(A)$. The only integers whose reciprocals are also integers are $\pm 1$. Therefore $\det(A) = \pm 1$. 

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(c) By part (a), if $A$ is a matrix with integer entries and $\det(A) = \pm 1$, then $A^{-1}$ has integer entries. By part (b) if $A$ and $A^{-1}$ both have integer entries then $\det(A) = \pm 1$. Therefore, if $A$ is an $n \times n$ matrix with integer entries, then $A^{-1}$ has integer entries if and only if $\det(A) = \pm 1$.

Remark: In proving the direction in (b), namely that if $A^{-1}$ has integer entries, we must have $\det(A) = \pm 1$, it’s tempting to also try and use the formula for the inverse in terms of the adjoint:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

and argue that if $\det(A)$ is not $\pm 1$ then this would introduce denominators and hence $A^{-1}$ could not have integer entries.

But there is a gap in this argument – just because we divide by something doesn’t mean that the result can’t be an integer, since there could be cancellation. For instance, $\frac{1}{3} \cdot 6 = 2$, and it’s certainly not true that “2 is not an integer” just because we saw the denominator 3.

The argument we gave that the determinant must be $\pm 1$ gives an indirect proof that not every entry of the adjoint can be divisible by $\det(A)$ (if $\det(A) \neq \pm 1$). It’s also possible to prove directly that there has to be at least one entry of the adjoint where the denominator of $\det(A)$ isn’t completely cancelled out, but it is trickier. The easiest way that I can think of uses “mod $p$” thinking, and in the end is largely (although in a slightly disguised way) equivalent to the previous argument using $\det(A^{-1}) = 1/\det(A)$.

3. Consider the following matrix $A$, and the transpose of its adjoint:

$$A = \begin{bmatrix} 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \text{adj}(A)^t = \begin{bmatrix} -3 & 3 & 3 & -3 \\ 0 & -2 & -2 & 3 \\ 3 & -5 & -2 & 3 \\ 0 & 3 & 0 & -3 \end{bmatrix}.$$

(a) Compute $A \cdot \text{adj}(A)$.

(b) What is $|A|$?

Answer the following questions by using the Laplace expansion formula to deduce how the determinant of $A$ changes when we change a single entry. When using the Laplace expansion formula, you will need to know the determinants of certain of $3 \times 3$ submatrices of $A$, but you can read that off from the matrix $\text{adj}(A)^t$ above. (Making this step slightly easier was the reason for writing the transpose of the adjoint above, instead of the adjoint.)
(c) If we add 3 to $a_{14}$, what is the determinant of the new matrix?

(d) Going back to the original matrix $A$, if we change $a_{34}$ from 2 to 4, what is the determinant this time?

(e) Suppose we want to change a single entry of $A$ by 1 to make the determinant of the new matrix equal to 2. What entry of $A$ could we change to do this? And to what number should we change it to?

Solution.

(a) We compute

$$A \cdot \text{adj}(A) = \begin{bmatrix} 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} -3 & 0 & 3 & 0 \\ 3 & -2 & -5 & 3 \\ 3 & -2 & -2 & 0 \\ -3 & 3 & 3 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}. $$

(b) Since $A \cdot \text{adj}(A) = \det(A) I_4$, we conclude from part (a) that $\det(A) = -3$.

(c) Using Laplace expansion along the first row, the determinant of $A$ is:

$$\det(A) = a_{11} |A_{11}| - a_{12} |A_{12}| + a_{13} |A_{13}| - a_{14} |A_{14}|. $$

Let $B$ be the matrix we get by adding 3 to $a_{14}$. All the entries of $B$ (except $a_{14}$) are the same as the entries of $A$, and so by Laplace expansion along the first row of $A$ we get

$$\det(B) = a_{11} |A_{11}| - a_{12} |A_{12}| + a_{12} |A_{13}| - (a_{14} + 3) |A_{14}|. $$

Comparing the two formulas gives

$$\det(B) = \det(A) - 3 |A_{14}|. $$

We could also arrive at this conclusion by Laplace expansion along the fourth column of $A$, or by using the multilinearity of the determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} + 3 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} & 3 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix} = \det(A) - 3 |A_{14}|. $$

In the last equality we have again used Laplace expansion down the fourth column of $A$. 

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We can read off $-|A_{14}|$ from $\text{adj}(A)^t$. From the formula for the adjoint we have

$$adj(A)^t = \begin{bmatrix}
|A_{11}| & -|A_{12}| & |A_{13}| & -|A_{14}|
-|A_{21}| & |A_{22}| & -|A_{23}| & |A_{24}|
|A_{31}| & -|A_{32}| & |A_{33}| & -|A_{34}|
-|A_{41}| & |A_{42}| & -|A_{43}| & |A_{44}|
\end{bmatrix} = \begin{bmatrix}
-3 & 3 & 3 & -3 \\
0 & -2 & -2 & 3 \\
3 & -5 & -2 & 3 \\
0 & 3 & 0 & -3 \\
\end{bmatrix}$$

and we conclude that $-|A_{14}| = -3$.

Therefore,

$$\det(B) = \det(A) + 3(-|A_{14}|) = -3 + 3(-3) = -3 + -9 = -12.$$

(d) Let $C$ be the matrix we get by changing $a_{34}$ from 2 to 4, i.e., by adding 2 to $a_{34}$. As in part (c) the Laplace expansion formula tells us that

$$\det(C) = \det(A) + 2(-|A_{34}|),$$

and from $\text{adj}(A)^t$ we see that $-|A_{34}| = 3$. Therefore

$$\det(C) = \det(A) + 2(-|A_{34}|) = -3 + 2 \cdot (3) = -3 + 6 = 3.$$

(e) Let $D$ be the matrix obtained from $A$ by changing a single entry, in position $(i, j)$ of $A$ by 1 (i.e., by adding $+1$ to $a_{ij}$). The Laplace expansion formula then tells us that

$$\det(D) = \det(A) \pm \left((-1)^{i+j}|A_{ij}|ight) = -3 \pm (-1)^{i+j}|A_{ij}|.$$ 

We want to find an entry $(i, j)$, and the choice of adding either $+1$ or $-1$ in order to make $\det(D) = 2$. From the formula above, this means that we want

$$2 = -3 \pm (-1)^{i+j}|A_{ij}|$$

or

$$5 = \pm (-1)^{i+j}|A_{ij}|.$$ 

The only minor of $A$ with absolute value 5 is $|A_{32}| = 5$, which comes with sign $(-1)^{3+2} = -1$. Therefore we want to choose the sign $\pm$ so that $5 = \pm(-5)$, which tells us that the sign should be negative. Therefore, in order to get a matrix whose determinant is 2, we should subtract 1 from $a_{34}$, changing it from a 2 to a 1.
4. For the basis $\mathcal{B} = [(3, 5), (1, 2)]$ in $\mathbb{R}^2$,

(a) Express $(4, 3), (1, 2),$ and $(1, 3)$ in $\mathcal{B}$ coordinates.

(b) Express $(2, 0)_{\mathcal{B}}, (1, 1)_{\mathcal{B}},$ and $(-1, 4)_{\mathcal{B}}$ in the standard coordinates.

Solutions.

(a) The matrix which converts from standard coordinates to $\mathcal{B}$-coordinates is

$$N^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}.$$ 

Since

$$\begin{bmatrix} 5 \\ -11 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix},$$

the vector $(4, 3)$ in usual coordinates is the same as $(5, -11)_{\mathcal{B}}$ in $\mathcal{B}$ coordinates.

We could also have worked this out by solving for $c_1$ and $c_2$ in the equation

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ 

The only solution is $c_1 = 5$ and $c_2 = -11$, so the $\mathcal{B}$ coordinates of the vector $(4, 3)$ are $(5, -11)_{\mathcal{B}}$.

Similarly, since

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

we have $(1, 2) = (0, 1)_{\mathcal{B}}$ and $(1, 3) = (-1, 4)_{\mathcal{B}}$. We could have seen the second formula without any calculation. Since $(1, 2)$ is the second basis vector in the basis $\mathcal{B}$, its $\mathcal{B}$-coordinate is necessarily $(0, 1)_{\mathcal{B}}$.

(b) This direction is even easier; the matrix changing from $\mathcal{B}$-coordinates to standard coordinates is $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$. 

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Since

\[
\begin{bmatrix} 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \\
\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix},
\]

we conclude that \((2, 0)_B = (6, 10)\), \((1, 1)_B = (4, 7)\), and \((-1, 4)_B = (1, 3)\). We also found the last equality in part (a), just the other way around.
Appendix: The Bayer Method.

To find the adjoint and inverse of \((3 \times 3)\)-matrices, do the following:

- Transpose the original matrix, leaving plenty of space;
- Copy over the first two columns, on the right;
- Copy over the first two rows, below;
- Cross out the first row and column;
- In the 9 spaces between the \((2 \times 2)\)-blocks, write each \((2 \times 2)\) determinant and circle it;
- Copy out these numbers in a matrix — this is the adjoint. To check your work, multiply by the original matrix. You should get the identity matrix times a number. This number is the determinant of the original matrix. Divide the adjoint by the determinant to get the inverse.

Example: If \(A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 1 & 4 & 5 \end{bmatrix}\) then we have:

\[
\begin{array}{cccc}
1 & 3 & 1 & 1 & 3 \\
2 & 1 & 4 & 2 & 1 \\
3 & 1 & 5 & 3 & 1 \\
1 & 3 & 1 & 1 & 3 \\
2 & 1 & 4 & 2 & 1 \\
\end{array}
\]

Since \( \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -14 & 2 & 8 \\ 11 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \) we conclude that \(A^{-1} = \frac{1}{6} \begin{bmatrix} 1 & 2 & 8 \\ -14 & 2 & -5 \end{bmatrix}\).

Alternatively, for \(3 \times 3\) matrices it is also easy to remember the formula for the adjoint in terms of the minors:

\[
\text{adj}(A) = \begin{bmatrix} |A_{11}| & -|A_{21}| & |A_{31}| \\ -|A_{12}| & |A_{22}| & -|A_{32}| \\ |A_{13}| & -|A_{23}| & |A_{33}| \end{bmatrix}.
\]

The Bayer method above is a way of writing out a grid of numbers so that the \((2 \times 2)\) determinants are exactly the \(|A_{ij}|\) with the correct signs.