1. Suppose that $A_n$ is the $n \times n$ matrix which has 2’s on the diagonal, and 1’s everywhere else:

\[
A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}, \ldots
\]

and suppose that $B_n$ is the $n \times n$ matrix which is just filled with minus-ones:

\[
B_2 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}, \ldots
\]

In this problem we will use some of our knowledge of characteristic polynomials to find a formula for $\det(A_n)$.

(a) Explain why $\det(A_n) = \det(I_n - B_n)$, where $I_n$ is the $n \times n$ identity matrix.

(b) If $P_n(t)$ is the characteristic polynomial of $B_n$, explain why $\det(A_n) = P_n(1)$.

This means that we can compute $\det(A_n)$ by first figuring out the characteristic polynomial of $B_n$ and then plugging in a value. It might seem like more work to compute the characteristic polynomial of $B_n$, but

(c) Since $B_n$ has rank 1, explain why this means that $t^{n-1}$ has to divide $P_n(t)$. (HINT: How big is the kernel of $B_n$? What is the relation between the kernel of $B_n$ and the eigenspace $E_0$ for $B_n$?)

This means that $P_n(t)$ is of the form $t^{n-1}(t - a)$ for some number $a$.

(d) Either by looking at the trace of $B_n$, or by seeing what happens to the vector $\vec{v} = (1, 1, \ldots, 1)$ of all 1’s when you put it through $B_n$, find the value of $a$.

(e) What is $\det(A_n)$?

(f) What is the determinant of the $n \times n$ matrix $C_n$ which has 5’s on the diagonal, and 1’s everywhere else?
Solution.

(a) Since $A_n = I_n - B_n$, it certainly has to be true that $\det(A_n) = \det(I_n - B_n)$.

(b) Since $P_n(t) = \det(tI_n - B_n)$ we have

$$P_n(1) = \det(1 \cdot I_n - B_n) = \det(I_n - B_n) = \det(A_n)$$

by part (a).

(c) The eigenspace $E_0$ is the same as the kernel of $B_n$. That comes directly from either of the definitions of $E_\lambda$ for any eigenvalue $\lambda$. For instance, since $E_\lambda = \text{Ker}(B_n - \lambda I_n)$, then when $\lambda = 0$ this gives us $E_0 = \text{Ker}(B_n - 0I_n) = \text{Ker}(B_n)$.

Since $B_n$ has rank 1, the rank-nullity theorem tells us that the kernel $E_0$ has dimension $n - 1$. In class on Thursday, March 2nd, we proved a proposition that (geometric multiplicity of $\lambda$) $\leq$ (algebraic multiplicity of $\lambda$) for each eigenvalue $\lambda$. Applying this proposition for $\lambda = 0$ we have:

$$n - 1 = \dim E_0 \leq (\text{multiplicity of the root } t = 0 \text{ in } P_n(t))$$

this means that $(t - 0)^{n-1} = t^{n-1}$ is a factor of $P_n(t)$.

(d) If we put the vector $\vec{v} = (1, 1, \ldots, 1)$ of all 1’s through $B_n$ we get the vector

$$B\vec{v} = (-n, -n, \ldots, -n) = -n\vec{v}$$

of all $-n$’s, so $-n$ is an eigenvalue of $B_n$, and $(t + n)$ is also a factor of $P_n(t)$.

Alternatively: the trace of $B_n$ is $-n$, which is the sum of all the eigenvalues, with multiplicity. Since at least $n-1$ of the eigenvalues are zero, the remaining unknown eigenvalue must be $-n$.

In either case, we’ve determined all the factors of $P_n(t)$, so we know what it is exactly: $P_n(t) = t^{n-1}(t + n)$.

(e) Combining parts (b) and (d):

$$\det(A_n) = P_n(1) = (1)^{n-1}(1 + n) = n + 1.$$ 

(f) The matrix being described is $4I_n - B_n$, so by the same reasoning as in part (b) the determinant of this matrix is

$$P_n(4) = (4)^{n-1}(4 + n) = 4^{n-1}(n + 4).$$
2. For the following three matrices, find their characteristic polynomials, the algebraic and geometric multiplicities of each eigenvalue, and a basis for each of their eigenspaces.

(a) \[
\begin{bmatrix}
2 & -1 & 2 \\
1 & 2 & 0 \\
0 & -1 & 4
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
4 & 6 & -2 \\
-1 & -1 & 1 \\
-1 & -3 & 3
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
-6 & 9 & 6 \\
0 & 3 & 0 \\
-12 & 12 & 11
\end{bmatrix}
\]

To make factoring the characteristic polynomials a bit easier, 2 is a root of each one.

(d) Which of the matrices above is diagonalizable?

(e) For each of the matrices \(A\) from part (d), find an invertible matrix \(N\) so that \(N^{-1}AN\) is a diagonal matrix.

Solution.

(a) The characteristic polynomial is \(t^3 - 8t^2 + 21t = 18 = (t - 2)(t - 3)^2\).

For \(\lambda = 2\) the eigenspace \(E_2\) is one-dimensional, spanned by \((0, 2, 1)\). To work out \(E_3\) we subtract 3 from the diagonals and row-reduce:

\[
\begin{bmatrix}
-1 & -1 & 2 \\
1 & -1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\text{RREF} \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}.
\]

Since this matrix has rank 2, its kernel (and hence \(E_3\)) is one dimensional. In this case it is spanned by \((1, 1, 1)\).

The geometric and algebraic multiplicities of \(\lambda = 2\) are both 1. The geometric multiplicity of \(\lambda = 3\) is 1, and the algebraic multiplicity is 2.

(b) The characteristic polynomial is \(t^3 - 6t^2 + 12t - 8 = (t - 2)^3\).

For \(\lambda = 2\) we subtract 2 from the diagonals and row reduce to get:

\[
\begin{bmatrix}
2 & 6 & -2 \\
-1 & -3 & 1 \\
-1 & -3 & 1
\end{bmatrix}
\text{RREF} \begin{bmatrix}
1 & 3 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

This matrix is of rank 1, so \(E_2\) (the kernel) is only 2 dimensional. One possible basis is \((3, -1, 0)\) and \((0, 1, 3)\).

Here the geometric multiplicity of the eigenvalue \(\lambda = 2\) is 2, and the algebraic multiplicity is 3.
(c) The characteristic polynomial is \( t^3 - 8t^2 + 21t - 18 = (l - 2)(l - 3)^2 \), just like in part (a).

For \( \lambda = 2 \) the eigenspace \( E_2 \) is one-dimensional, spanned by \((3, 0, 4)\). To work out \( E_3 \) we again subtract 3 from the diagonals and row-reduce:

\[
\begin{bmatrix}
-9 & 9 & 6 \\
0 & 0 & 0 \\
-12 & 12 & 8
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & -1 & -2/3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

This time the matrix is of rank 1 so we know that \( E_3 \) will be 2 dimensional. One possible basis is \((1, 1, 0)\) and \((2, 0, 3)\).

The geometric and algebraic multiplicities of \( \lambda = 2 \) are both 1. The geometric and algebraic multiplicities of \( \lambda = 3 \) are both 2.

(d) Only the matrix from part (c) is diagonalizable. It is the only matrix where the algebraic and geometric multiplicities are the same for all eigenvalues. In other words, it is the only matrix for which we will be able to find enough eigenvectors to form a basis of \( \mathbb{R}^3 \).

(e) By part (d), matrix from (c) is the only one which can be diagonalized. Let \( C \) be this matrix. To diagonalize \( C \) we need a basis of eigenvectors for \( C \). The vectors \( \vec{v}_1 = (3, 0, 4) \), \( \vec{v}_2 = (1, 1, 0) \), and \( \vec{v}_3 = (2, 0, 3) \) will work. \((\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_3) \) a basis for \( E_3 \). In the proof of the theorem that “if the algebraic and geometric multiplicities are equal, the matrix is diagonalizable” we also proved that taking a basis for each eigenspace and putting the bases together in one big list gave a basis for \( \mathbb{R}^n \).

With respect to the basis \( B = [\vec{v}_1, \vec{v}_2, \vec{v}_3] \), the change of basis matrix is

\[
N = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 4 & 0 & 3 \end{bmatrix},
\]

and

\[
N^{-1}CN = \begin{bmatrix} 3 & -3 & -2 \\ 0 & 1 & 0 \\ -4 & 4 & 3 \end{bmatrix} \begin{bmatrix} -6 & 9 & 6 \\ 0 & 3 & 0 \\ -12 & 12 & 11 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ \vec{v} & 0 & 3 \\ \vec{v} \end{bmatrix}.
\]

4
3. Suppose that $B$ and $C$ are $n \times n$ matrices, and that $N$ is an invertible $n \times n$ matrix so that $C = N^{-1}BN$.

(a) Find a formula expressing $B$ in terms of $N$ and $C$ (i.e., “Solve for $B$”).

(b) Show that $C^2 = N^{-1}B^2N$. (HINT: just multiply.)

(c) Suppose that $D$ is a diagonal matrix with real entries. If we want to find a diagonal matrix $C$ with real entries, such that $C^2 = D$, what has to be true about the eigenvalues of $D$?

(d) Let $A = \begin{bmatrix} -16 & -10 \\ 50 & 29 \end{bmatrix}$.

Find a real matrix $B$ with $B^2 = A$ (i.e., a “square root” of $A$).

**Solution.**

(a) Starting with $C = N^{-1}BN$, we multiply on the left by $N$ to get

$$NC = NN^{-1}BN = (NN^{-1})BN = IBN = BN.$$  

We then multiply on the right by $N^{-1}$ to get

$$NCN^{-1} = BNN^{-1} = B(NN^{-1}) = BI = B.$$  

Therefore $B = NCN^{-1}$.

(b) We have

$$C^2 = CC = (N^{-1}BN)(N^{-1}BN) = N^{-1}B(NN^{-1})BN = N^{-1}BIBN = N^{-1}BBN = N^{-1}B^2N.$$  

(c) Let

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

be an $n \times n$ diagonal matrix. We are looking for another diagonal matrix, say

$$F = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_n \end{bmatrix}$$
so that \( F^2 = D \). Computing \( F^2 \), this would mean that

\[
F^2 = \begin{bmatrix}
\alpha_1^2 & 0 & 0 & \cdots & 0 \\
0 & \alpha_2^2 & 0 & \cdots & 0 \\
0 & 0 & \alpha_3^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \alpha_n^2
\end{bmatrix} = \begin{bmatrix}
\lambda_1 & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots & 0 \\
0 & 0 & \lambda_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_n
\end{bmatrix} = D,
\]

so that \( \lambda_i = \alpha_i^2 \) for \( i = 1, \ldots, n \). If \( \alpha_1, \ldots, \alpha_n \) are real numbers, this implies that \( \lambda_i \geq 0 \) for \( i = 1, \ldots, n \). On the other hand, if each \( \lambda_i \geq 0 \) then such a matrix \( F \) exists: take \( \alpha_i = \sqrt{\lambda_i} \) for \( i = 1, \ldots, n \).

(d) Let us first look for a change of basis to make \( A \) diagonal. The characteristic polynomial of \( A \) is \( t^2 - 13t + 36 = (t - 4)(t - 9) \).

\[ \lambda = 4 \] : 
\[
A - 4I_2 = \begin{bmatrix}
-20 & -10 \\
50 & -25
\end{bmatrix} \implies \begin{bmatrix} 1 & \frac{1}{2} \\
0 & 0
\end{bmatrix},
\]
whose kernel has basis \( \vec{v}_1 = (1, -2) \).

\[ \lambda = 9 \] : 
\[
A - 9I_2 = \begin{bmatrix}
-25 & -10 \\
50 & -20
\end{bmatrix} \implies \begin{bmatrix} 1 & \frac{2}{5} \\
0 & 0
\end{bmatrix},
\]
whose kernel has basis \( \vec{v}_2 = (2, -5) \).

The basis \( \mathcal{B} = [\vec{v}_1, \vec{v}_2] \) is therefore a basis of eigenvectors of \( A \). Using the change of basis matrix

\[
N = \begin{bmatrix} 1 & 2 \\
-2 & -5
\end{bmatrix}
\]
with inverse \( N^{-1} = \begin{bmatrix} 5 & 2 \\
-2 & -1
\end{bmatrix} \),

we have

\[
N^{-1}AN = \begin{bmatrix} 5 & 2 \\
-2 & -1
\end{bmatrix} \begin{bmatrix}
-16 & -10 \\
50 & 29
\end{bmatrix} \begin{bmatrix} 1 & 2 \\
-2 & -5
\end{bmatrix} = \begin{bmatrix} 4 & 0 \\
0 & 9
\end{bmatrix}.
\]

Let us call this last diagonal matrix \( D \). One possible square root for \( D \) is (as mentioned in (c)), the diagonal matrix

\[
F = \begin{bmatrix} 2 & 0 \\
0 & 3
\end{bmatrix}.
\]
Changing basis the other direction, we get

\[ NFN^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 10 & 7 \end{bmatrix}, \]

which is a square root of \( A \).

It is good to actually check that this matrix is a square root:

\[ \begin{bmatrix} -2 & -2 \\ 10 & 7 \end{bmatrix}^2 = \begin{bmatrix} -2 & -2 \\ 10 & 7 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 10 & 7 \end{bmatrix} = \begin{bmatrix} -16 & -10 \\ 50 & 29 \end{bmatrix}. \]

**Remark.** There are three other possible square roots of \( A \), coming from the three other possible square roots of the diagonal matrix \( D \). These other square roots are:

\[ \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -22 & -10 \\ 50 & 23 \end{bmatrix}, \]

\[ \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 22 & 10 \\ -50 & -23 \end{bmatrix}, \]

and

\[ \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -10 & -7 \end{bmatrix}. \]

(The matrices to the left and right of the diagonal matrices are, as before, \( N \) and \( N^{-1} \) respectively.)