Math 110 — Homework Assignment 9

DUE DATE: NOV. 18, 2016

1.

(a) Prove or disprove: the vectors $\vec{v}_1 = (2, 3)$ and $\vec{v}_2 = (1, 5)$ form a basis for $\mathbb{R}^2$.

(b) Prove or disprove: the vectors $\vec{w}_1 = (1, 9)$ and $\vec{w}_2 = (2, 3)$ form a basis for $\mathbb{R}^2$.

(c) How many bases can a subspace have?

(d) Let V be the set of vectors (x, y, z, w) in $\mathbb{R}^4$ which are the solutions to the equations:

$$\begin{aligned} x + 0y + 3z - 2w &= 0 \\ 0x + y - 4z - 9w &= 0 \end{aligned}$$

The subset V is a subspace of $\mathbb{R}^4$. Find a basis for V.

(SUGGESTION: V is given as the set of solutions to a system of linear equations. You know how to parameterize all the solutions...)

2. Let $\vec{v}_1, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$, and let $A = [\vec{v}_1 \mid \vec{v}_2 \mid \cdots \vec{v}_k \quad]$, i.e., the matrix whose columns are $\vec{v}_1, \ldots, \vec{v}_k$. Prove that $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent if and only if $\mathrm{Rank}(A) = k$.

REMINDER: Proving a statement with an "if and only if" requires proving both directions. You assume that $\mathrm{Rank}(A) = k$ and then deduce that $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent. Then assume that $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent and prove that $\mathrm{Rank}(A) = k$. (If you can do both steps at the same time that is fine too.) One other reminder: looking for $c_1, \ldots, c_k$ so that $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ is the same as solving a system of linear equations.

3. *Linear transformation puzzlers*

(a) Consider a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. If $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent vectors in $\mathbb{R}^n$, are the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_k)$ necessarily linearly dependent in $\mathbb{R}^m$? If so, why?

(b) If A is an $n \times p$ matrix, and B is a $p \times m$ matrix, with $\mathrm{Im}(B) \subseteq \mathrm{Ker}(A)$, what can you say about the product AB?

(c) if A is a $p \times m$ matrix, and B a $q \times m$ matrix, and we make a $(p+q) \times m$ matrix C by "stacking" A on top of B:

$$C = \begin{bmatrix} A \\ B \end{bmatrix},$$

what is the relation between $\mathrm{Ker}(A)$, $\mathrm{Ker}(B)$, and $\mathrm{Ker}(C)$?

Note: So far, we have only used the symbols Ker and Im when talking about a linear transformation T. In this homework we're going to extend this notation and also use Ker and Im when talking about a matrix. If A is an $m \times n$ matrix, then

$$\mathrm{Ker}(A) = \left\{ \vec{v} \in \mathbb{R}^n \mid A\vec{v} = \vec{0} \right\}.$$

While

$$\mathrm{Im}(A) = \left\{ \vec{w} \in \mathbb{R}^m \mid \text{there is a } v \in \mathbb{R}^n \text{ so that } A\vec{v} = \vec{w} \right\}.$$

The connection between this notation and our usual notation about linear transformations is that if $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation and A the standard matrix of T, then $\mathrm{Ker}(T) = \mathrm{Ker}(A)$ and $\mathrm{Im}(T) = \mathrm{Im}(A)$.

4.

(a) Suppose that we have a system of linear equations in n variables. For instance, we might have m equations:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{k2}x_2 + \cdots + a_{mn}x_n & = & 0 \end{array}$$

where the a_{ij} are any numbers in $\mathbb{R}$. Show that the set of solutions to this system of equations forms a subspace of $\mathbb{R}^n$.

(b) The vectors $\vec{v}_1 = (-1,3,1,2)$, $\vec{v}_2 = (2,3,2,-7)$, and $\vec{v}_3 = (2,1,1,-6)$ span a 3-dimensional subspace of $\mathbb{R}^4$. Find a single equation of the form $ax+by+cz+dw = 0$ whose solutions are this subspace.