1. 

(a) Prove or disprove: the vectors $\vec{v}_1 = (2, 3)$ and $\vec{v}_2 = (1, 5)$ form a basis for $\mathbb{R}^2$. 

(b) Prove or disprove: the vectors $\vec{w}_1 = (1, 9)$ and $\vec{w}_2 = (2, 3)$ form a basis for $\mathbb{R}^2$. 

(c) How many bases can a subspace have? 

(d) Let $V$ be the set of vectors $(x, y, z, w)$ in $\mathbb{R}^4$ which are the solutions to the equations:

\[
\begin{align*}
  x + 0y + 3z - 2w & = 0 \\
  0x + y - 4z - 9w & = 0
\end{align*}
\]

The subset $V$ is a subspace of $\mathbb{R}^4$. Find a basis for $V$.

(Suggestion: $V$ is given as the set of solutions to a system of linear equations. You know how to parameterize all the solutions...) 

2. Let $\vec{v}_1, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$, and let $A = [\vec{v}_1 | \vec{v}_2 | \cdots | \vec{v}_k ]$, i.e., the matrix whose columns are $\vec{v}_1, \ldots, \vec{v}_k$. Prove that $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent if and only if $\text{Rank}(A) = k$. 

Reminder: Proving a statement with an “if and only if” requires proving both directions. You assume that $\text{Rank}(A) = k$ and then deduce that $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent. Then assume that $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent and prove that $\text{Rank}(A) = k$. (If you can do both steps at the same time that is fine too.) One other reminder: looking for $c_1, \ldots, c_k$ so that $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ is the same as solving a system of linear equations. 

3. Linear transformation puzzlers 

(a) Consider a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$. If $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent vectors in $\mathbb{R}^n$, are the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_k)$ necessarily linearly dependent in $\mathbb{R}^m$? If so, why? 

(b) If $A$ is an $n \times p$ matrix, and $B$ is a $p \times m$ matrix, with $\text{Im}(B) \subseteq \text{Ker}(A)$, what can you say about the product $AB$?
(c) If $A$ is a $p \times m$ matrix, and $B$ a $q \times m$ matrix, and we make a $(p + q) \times m$ matrix $C$ by “stacking” $A$ on top of $B$:

$$C = \begin{bmatrix} A \\ B \end{bmatrix},$$

what is the relation between $\text{Ker}(A)$, $\text{Ker}(B)$, and $\text{Ker}(C)$?

**Note:** So far, we have only used the symbols $\text{Ker}$ and $\text{Im}$ when talking about a linear transformation $T$. In this homework we’re going to extend this notation and also use $\text{Ker}$ and $\text{Im}$ when talking about a matrix. If $A$ is an $m \times n$ matrix, then

$$\text{Ker}(A) = \left\{ \vec{v} \in \mathbb{R}^n \mid A\vec{v} = \vec{0} \right\}.$$ 

While

$$\text{Im}(A) = \left\{ \vec{w} \in \mathbb{R}^m \mid \text{there is a } \vec{v} \in \mathbb{R}^n \text{ so that } A\vec{v} = \vec{w} \right\}.$$ 

The connection between this notation and our usual notation about linear transformations is that if $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation and $A$ the standard matrix of $T$, then $\text{Ker}(T) = \text{Ker}(A)$ and $\text{Im}(T) = \text{Im}(A)$.

4.

(a) Suppose that we have a system of linear equations in $n$ variables. For instance, we might have $m$ equations:

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{k2}x_2 + \cdots + a_{mn}x_n &= 0
\end{align*}
\]

where the $a_{ij}$ are any numbers in $\mathbb{R}$. Show that the set of solutions to this system of equations forms a subspace of $\mathbb{R}^n$.

(b) The vectors $\vec{v}_1 = (-1, 3, 1, 2)$, $\vec{v}_2 = (2, 3, 2, -7)$, and $\vec{v}_3 = (2, 1, 1, -6)$ span a 3-dimensional subspace of $\mathbb{R}^4$. Find a single equation of the form $ax + by + cz + dw = 0$ whose solutions are this subspace.