Math 110 Homework Assignment 12

DUE DATE: DEC. 9, 2016

1. Let $\vec{v}_1 = (1,1,1,1)$ and $\vec{v}_2 = (3,5,2,1)$, and let V be the subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1$ and $\vec{v}_2$.

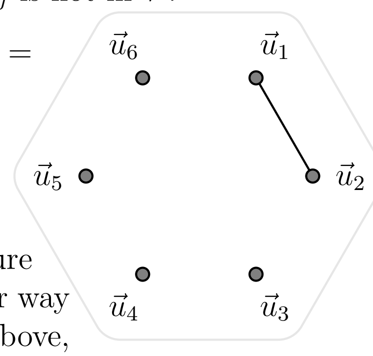
(a) Find two equations, each of the form $ax+by+cz+dw=0$, such that the common solution to these two equations is the subspace V.

(b) Find a linear transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ whose kernel is exactly V.

Let $\vec{u}_1 = (-3,4,-2,1)$, $\vec{u}_2 = (-6,-3,-3,2)$, $\vec{u}_3 = (-1,0,3,9)$, $\vec{u}_4 = (1,4,4,9)$, $\vec{u}_5 = (5,4,3,1)$, and $\vec{u}_6 = (5,2,4,3)$, all vectors in $\mathbb{R}^4$.

(c) On a copy of the diagram below, connect the dot labeled $\vec{u}_i$ to the dot labelled $\vec{u}_j$ with a line if $\vec{u}_i - \vec{u}_j$ is in V, and don't put any line if $\vec{u}_i - \vec{u}_j$ is not in V.

For example, there is a line connecting $\vec{u}_1$ and $\vec{u}_2$ since $\vec{u}_1 - \vec{u}_2 = (3,7,1,-1) = -3\vec{v}_1 + 2\vec{v}_2$ is in V, but there should be no line connecting $\vec{u}_1$ and $\vec{u}_6$ since $\vec{u}_1 - \vec{u}_6 = (-8,2,-6,-2)$ is not in V.



Note: A painful way to answer this question would be to take all 15 possible pairs of vectors $\vec{u}_i$, $\vec{u}_j$, subtract them, and for each one figure out if the difference is in V by using RREF. There is a much simpler way to answer this question using your linear transformation from (b) above, and the kernel principle.

(d) To check that things worked out as expected, take two of the points (other than $\vec{u}_1$ and $\vec{u}_2$) you connected by a line above, subtract the vectors, and check that the difference really is in V. Then take take two points not connected by a line and show that the difference is not in V.

2. Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation.

(a) If $\vec{x}$ is a vector in $\mathbb{R}^n$, and $\vec{v}$ a vector in the kernel of T, explain why $T(\vec{x}) = T(\vec{x}+\vec{v})$.

(b) Conversely, if $\vec{x}_1$ and $\vec{x}_2$ are vectors in $\mathbb{R}^n$, and if $T(\vec{x}_1) = T(\vec{x}_2)$, explain why there is a vector $\vec{v}$ in the kernel of T with $\vec{x}_2 = \vec{x}_1 + \vec{v}$. (HINT: What does T do to $\vec{x}_2 - \vec{x}_1$?)

(c) If $\vec{w}$ is a vector in $\mathbb{R}^m$, and $\vec{x}_1$ a vector in $\mathbb{R}^n$ with $T(\vec{x}_1) = \vec{w}$, explain why the solutions to the equation $T(\vec{x}) = \vec{w}$ are all of the form $\vec{x} = \vec{x}_1 + \vec{v}$, with $\vec{v}$ in $\text{Ker}(T)$.

3. In class, when we were studying the example $T(x, y, z) = (z, z)$ it was convenient to factor the linear transformation into a map $T_1 \colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ followed by a map $T_2 \colon \mathbb{R} \longrightarrow \mathbb{R}^2$ such that T_1 was surjective, T_2 injective, and $T = T_2 \circ T_1$.

It turns out that something like this is always possible. That is, given a linear transformation $T \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$, it is always possible to find linear transformations $T_1 \colon \mathbb{R}^n \longrightarrow \mathbb{R}^p$ and $T_2 \colon \mathbb{R}^p \longrightarrow \mathbb{R}^m$, with $p = \dim(\mathrm{Im}(T))$, such that T_1 is surjective, T_2 injective, and $T = T_2 \circ T_1$. In this problem we will construct such a "factorization of T" in an example, although the method works for any T.

Let $T \colon \mathbb{R}^6 \longrightarrow \mathbb{R}^4$ be the linear transformation with standard matrix

$$A = \begin{bmatrix} 2 & 5 & -1 & 4 & 5 & -4 \\ -1 & 3 & -5 & 1 & 8 & 1 \\ 2 & 3 & 1 & 2 & 3 & 0 \\ 2 & 1 & 3 & 1 & -1 & 0 \end{bmatrix} \text{ which has RREF } \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Let $p = \dim(\mathrm{Im}(T))$. Explain how you know that $p = 3$.

(b) Let B be the 3×6 matrix obtained from $\mathrm{RREF}(A)$ by chopping off all the zero rows. (In this case there is only one zero row). The matrix B defines a linear transformation $T_1 \colon \mathbb{R}^6 \longrightarrow \mathbb{R}^3$. Explain how you know that this linear transformation is surjective.

(c) Find a basis for $\mathrm{Im}(T)$.

(d) Let C be the 4×3 matrix whose columns are the vectors from part (c). The matrix C defines a linear transformation $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$. Show that this linear transformation is injective.

To complete the construction, we need to show that $T = T_2 \circ T_1$. We can compute the standard matrix of $T_2 \circ T_1$ by computing CB.

(e) Compute CB.

(f) Is $T = T_2 \circ T_1$? How do you know?

Note: This assignment is due on or before **Friday, December 9th, at 5pm**. The assigment can be handed in to my office, 507 Jeffery Hall. (There is a mailbox with my name on it to the right of my door. There is an even closer mailbox with Francesco Cellarosi's name on it to the left of my door. Do not be tempted by the closer mailbox!)