1. Let $A$ be the matrix

$$A = \begin{bmatrix} 3 & 2 & z \\ x & 1 & 3 \\ 2 & y & 5 \end{bmatrix},$$

where $x$, $y$, and $z$ are variables.

(a) Compute the adjoint matrix of $A$ (this will still involve the variables $x$, $y$, and $z$).

(b) Compute the product of the adjoint matrix and $A$.

(c) Compute $\det(A)$.

(d) Assuming that $\det(A) \neq 0$, write down the inverse of $A$ (this will still be a matrix involving $x$, $y$, and $z$).

(e) To show that this method really gives a “universal formula for the inverse”, plug in the values $(x, y, z) = (3, -1, 1)$ into both $A$ and the inverse matrix from part (d), and multiply them to see that it really gives the inverse for $A$.

(f) Do the same for $(x, y, z) = (1, 1, 1)$

2. Suppose that $A$ is an $n \times n$ matrix with integer entries, and that $A$ is invertible. Since $A$ is invertible, we can compute the matrix $A^{-1}$. The computations seem much cleaner (and friendlier) when $A^{-1}$ also has only integer entries. The purpose of this question is to figure out when that happens.

(a) Show that if $\det(A) = \pm 1$ then $A^{-1}$ has only integer entries. (Suggestion: Use the expression for the inverse in terms of the adjoint matrix.)

(b) Conversely suppose that $A^{-1}$ also has integer entries. Using the fact that

$$\det(A) \det(A^{-1}) = \det(I_n) = 1,$$

explain why we must have $\det(A) = \pm 1$.

(c) Conclude that if $A$ is a square matrix with integer entries then $A^{-1}$ has integer entries if and only if $\det(A) = \pm 1$. 
3. Consider the following matrix $A$, and the transpose of its adjoint:

$$A = \begin{bmatrix} 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \text{ and } \text{adj}(A)^t = \begin{bmatrix} -3 & 3 & 3 & -3 \\ 0 & -2 & -2 & 3 \\ 3 & -5 & -2 & 3 \\ 0 & 3 & 0 & -3 \end{bmatrix}.$$ 

(a) Compute $A \cdot \text{adj}(A)$. 

(b) What is $|A|$?

Answer the following questions by using the Laplace expansion formula to deduce how the determinant of $A$ changes when we change a single entry. When using the Laplace expansion formula, you will need to know the determinants of certain of $3 \times 3$ submatrices of $A$, but you can read that off from the matrix adj$(A)^t$ above. (Making this step slightly easier was the reason for writing the transpose of the adjoint above, instead of the adjoint.)

(c) If we add 3 to $a_{14}$, what is the determinant of the new matrix? 

(d) Going back to the original matrix $A$, if we change $a_{34}$ from 2 to 4, what is the determinant this time? 

(e) Suppose we want to change a single entry of $A$ by 1 to make the determinant of the new matrix equal to 2. What entry of $A$ could we change to do this? And to what number should we change it to? 

4. For the basis $\mathcal{B} = [(3, 5), (1, 2)]$ in $\mathbb{R}^2$, 

(a) Express $(4, 3)$, $(1, 2)$, and $(1, 3)$ in $\mathcal{B}$-coordinates. 

(b) Express $(2, 0)_B$, $(1, 1)_B$, and $(-1, 4)_B$ in the standard coordinates.