1. Suppose that \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) is given (in the standard coordinates) by the matrix
   \[
   A = \begin{bmatrix}
   2 & 1 & 5 \\
   1 & 1 & 3
   \end{bmatrix}.
   \]

   Let \( B \) be the basis \( B = [(1, 1, 1), (2, 0, 1), (3, 2, 1)] \) of \( \mathbb{R}^3 \), and let \( A \) be the basis \( A = [(3, 5), (1, 2)] \) of \( \mathbb{R}^2 \), find the matrix for \( T \) with respect to the new basis on both sides.

2. Let \( \vec{v}_1 = (1, 2), \vec{v}_2 = (2, -1) \), and let \( B \) be the basis \( B = [\vec{v}_1, \vec{v}_2] \) of \( \mathbb{R}^2 \).

   Let \( T \) be the linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) given by \( T(\vec{v}_1) = \vec{v}_1 \) and \( T(\vec{v}_2) = \vec{0} \).

   (a) Write down the matrix for \( T \) in the new basis \( B \). (You should be able to do this directly from the definition of \( T \) and the definition of “writing the matrix of a linear transformation with respect to a basis”).

   (b) Use this to write down the matrix for \( T \) in the standard basis.

   You might want to compare the answer for (b) with the answer for H6 3(a), with \( m = 2 \). Can you see why these are the same?

   \textbf{Note:} In part (b) you need to go from the matrix in \( B \)-basis form to the matrix in standard basis form, which is the reverse of what we did in class, so think for a bit to figure out which way the change of basis matrices should go.

3. We’ll check in class that the determinant of a square matrix doesn’t change when we change basis. The purpose of this question is to show that the \textit{trace} of a matrix also doesn’t change when we change basis.

   For an \( n \times n \) matrix \( A \), the \textit{trace} of \( A \), \( \text{tr}(A) \) is the sum of the numbers on the diagonal. For instance, if
   \[
   A = \begin{bmatrix}
   1 & 3 & 5 \\
   2 & 7 & 9 \\
   6 & 0 & 4
   \end{bmatrix}
   \]
   then \( \text{tr}(A) = 1 + 7 + 4 = 12 \).

   In the \( a_{ij} \) notation for the entries of a matrix, \( \text{tr}(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn} \).

   (a) If \( A \) and \( B \) are \( n \times n \) matrices, prove that \( \text{tr}(AB) = \text{tr}(BA) \) (the formula in the book for \( ij \)-th entry for a product of matrices may help).
(b) Suppose that $A$ is an $n \times m$ matrix and $B$ an $m \times n$ matrix. Then both products $AB$ (an $n \times n$ matrix) and $BA$ (an $m \times m$ matrix) are square matrices, so we can take their traces. PROVE OR DISPROVE: $\text{tr}(AB) = \text{tr}(BA)$ in this case.

(c) If $C$ is an $n \times n$ matrix, and $M$ an invertible $n \times n$ matrix, prove that

$$\text{tr}(C) = \text{tr}(M^{-1}CM).$$

SUGGESTION: If you make the right choice of matrices $A$ and $B$, part (c) will follow from part (a) with very little work.

4. Suppose that $x_1, x_2, \ldots, x_n$ are numbers. The $n \times n$ matrix

$$A = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
x_1 & x_2 & x_3 & \cdots & x_n \\
x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\
x_1^3 & x_2^3 & x_3^3 & \cdots & x_n^3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1}
\end{bmatrix}$$

is called the Vandermonde matrix, and is surprisingly useful to know a few basic facts about it. Let’s establish one of them now.

If any of the two $x_i$’s are equal to each other, then of course $\det(A) = 0$ since we will have two repeated columns. What we’d like to show is that if all of the $x_i$’s are different, then $\det(A) \neq 0$, i.e., that $A$ is an invertible matrix.

(a) Explain why showing that $\det(A) \neq 0$ is the same as showing that $\det(A^t) \neq 0$, where $A^t$ means the transpose of $A$. [This is a very short answer].

(b) Suppose that $A^t$ is invertible. Explain why this means that $\text{Ker}(A^t) = \{\vec{0}\}$.

(c) Conversely, suppose that $\text{Ker}(A^t) = \{\vec{0}\}$. Explain why this means that $A^t$ is invertible. (SUGGESTION: The rank-nullity theorem may help at one point.)

(d) Explain why showing that $\det(A^t) \neq 0$ is the same as showing that the only vector in $\text{Ker}(A^t)$ is the zero vector.

(e) If $\vec{v} = (c_0, c_1, \ldots, c_{n-1})$ is a vector in the kernel of $A^t$, and $\vec{v}$ is not the zero vector, explain how this would give you a polynomial of degree $\leq n - 1$ with more than $n - 1$ roots, which would be a contradiction. [HINT: write out what $A^t\vec{v} = \vec{0}$ means.] Make sure that you answer carefully, for instance, in your explanation why is it important that all the $x_i$’s be different?