1. Suppose that $A_n$ is the $n \times n$ matrix which has 2’s on the diagonal, and 1’s everywhere else:

$$A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}, \ldots$$

and suppose that $B_n$ is the $n \times n$ matrix which is just filled with minus-ones:

$$B_2 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}, \ldots$$

In this problem we will use some of our knowledge of characteristic polynomials to find a formula for $\det(A_n)$.

(a) Explain why $\det(A_n) = \det(I_n - B_n)$, where $I_n$ is the $n \times n$ identity matrix.

(b) If $P_n(t)$ is the characteristic polynomial of $B_n$, explain why $\det(A_n) = P_n(1)$.

This means that we can compute $\det(A_n)$ by first figuring out the characteristic polynomial of $B_n$ and then plugging in a value. It might seem like more work to compute the characteristic polynomial of $B_n$, but

(c) Since $B_n$ has rank 1, explain why this means that $t^{n-1}$ has to divide $P_n(t)$. (HINT: How big is the kernel of $B_n$? What is the relation between the kernel of $B_n$ and the eigenspace $E_0$ for $B_n$?).

This means that $P_n(t)$ is of the form $t^{n-1}(t - a)$ for some number $a$.

(d) Either by looking at the trace of $B_n$, or by seeing what happens to the vector $\vec{v} = (1, 1, \ldots, 1)$ of all 1’s when you put it through $B_n$, find the value of $a$.

(e) What is $\det(A_n)$?

(f) What is the determinant of the $n \times n$ matrix $C_n$ which has 5’s on the diagonal, and 1’s everywhere else?
2. For the following three matrices, find their characteristic polynomials, the algebraic and geometric multiplicities of each eigenvalue, and a basis for each of their eigenspaces.

\[
(a) \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix} \quad (b) \begin{bmatrix} 4 & 6 & -2 \\ -1 & -1 & 1 \\ -1 & -3 & 3 \end{bmatrix} \quad (c) \begin{bmatrix} -6 & 9 & 6 \\ 0 & 3 & 0 \\ -12 & 12 & 11 \end{bmatrix}
\]

To make factoring the characteristic polynomials a bit easier, 2 is a root of each one.

(d) Which of the matrices above is diagonalizable?

(e) For each of the matrices \( A \) from part (d), find an invertible matrix \( N \) so that \( N^{-1}AN \) is a diagonal matrix.

3. Suppose that \( B \) and \( C \) are \( n \times n \) matrices, and that \( N \) is an invertible \( n \times n \) matrix so that \( C = N^{-1}BN \).

(a) Find a formula expressing \( B \) in terms of \( N \) and \( C \) (i.e., “Solve for \( B \)).

(b) Show that \( C^2 = N^{-1}B^2N \). (HINT: just multiply.)

(c) Suppose that \( D \) is a diagonal matrix with real entries. If we want to find a diagonal matrix \( C \) with real entries, such that \( C^2 = D \), what has to be true about the eigenvalues of \( D \)?

(d) Let \( A = \begin{bmatrix} -16 & -10 \\ 50 & 29 \end{bmatrix} \).

Find a real matrix \( B \) with \( B^2 = A \) (i.e., a “square root” of \( A \)).