1. (a) If a is rational and b irrational, then $a + b$ is always irrational.

The most convincing way to make the argument is to use "proof by contradiction" — assume that the sum is rational and show how this leads to a contradiction. The argument goes like this:

Claim: If a is rational and b is irrational, then $a + b$ is always irrational.

Proof: Suppose that $a + b$ were rational, then we could write $a + b$ as the ratio of two integers, say $a + b = p/q$ with both p and q integers. Since a is rational we can also write a as $a = m/n$ with both m and n integers.

But then

$$b = (a + b) - a = \frac{p}{q} - \frac{m}{n} = \frac{pn - qm}{qn}$$

and so b would be a quotient of the integers $pn - qm$ and qn, i.e., b would be rational. Since this contradicts the original assumption that b is irrational, it can't be true that $a + b$ is rational, so the sum must always be irrational. $\square$

On the other hand, if a and b are both irrational, there is no way to know what type the sum $a + b$ will be. For instance, if $a = \sqrt{2}$ and $b = \sqrt{2}$, then $a + b = 2\sqrt{2}$ which is irrational (by part (b) below). But, if $a = \sqrt{2}$ and $b = 3 - \sqrt{2}$ then $a + b = 3$ is rational.

I.e., we have an example where a and b are both irrational and the sum is also irrational, and an example where a and b are both irrational and the sum is rational, so certainly knowing that a and b are irrational doesn't let us conclude anything about the type of the sum.

Note that this part of the argument was a little different from the first. In the first we were claiming that something was always true (that $a + b$ would always be irrational if a is rational and b irrational), and the second part we were claiming that in general its impossible to tell what happens if a and b are both irrational.

In the first part we had to give a general argument that would work for all rational a and irrational b, and in the second it was enough just to give two examples.

(b) This is almost exactly like the first half of part (a), with a small difference. If a is rational and b irrational, then there are two possibilities for the product ab:

(*i*) If $a \neq 0$ then the product ab is irrational, but

(*ii*) If $a = 0$ then the product ab is zero, which is a rational number.

The argument for part (ii) is easy: if $a = 0$ then $ab = 0$ and 0 is a rational number. To show (i) we repeat what we did above, but using multiplication in place of addition and division in place of subtraction.

Claim: If a is rational and $a \neq 0$, and if b is irrational, then the product ab is always irrational.

Proof: If the product ab were rational, we would be able to write the product ab as the ratio of two integers, say $ab = p/q$ with both p and q integers. Since a is rational we can write a as the quotient $a = m/n$ of two integers. Since $a \neq 0$ we can divide by a. But then

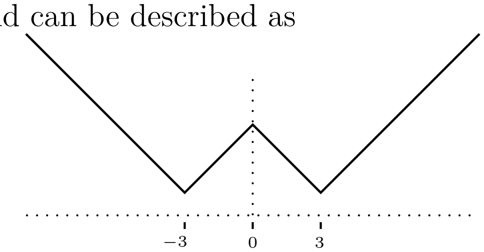
$$b = ab \cdot \frac{1}{a} = \frac{p}{q} \cdot \frac{n}{m} = \frac{pn}{qm}$$

and so b would be rational. This contradicts our original assumption about b, and so the product ab must be irrational. $\square$

(c) The answer is yes, it is possible to have such a number, and to justify our answer we just have to give a single example. One possible example is to pick $a = \sqrt[4]{2}$, since then $a^2 = \sqrt{2}$, which we know is irrational, and $a^4 = 2$ which is certainly rational. Many other examples are possible.

2. (a) the function $f(|x|)$ looks like a "W" and can be described as

$$f(|x|) = \begin{cases} x-2 & \text{if } x \geqslant 3 \\ 4-x & \text{if } 0 \leqslant x \leqslant 3 \\ x+4 & \text{if } -3 \leqslant x \leqslant 0 \\ -2-x & \text{if } x \leqslant -3 \end{cases}$$

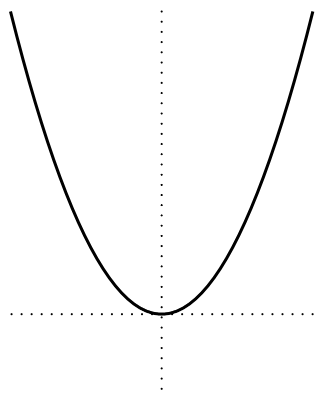


There are several ways to see this. One way is to simply consider the possibilities for x and how they effect the absolute value signs. For instance, if $x \geqslant 0$ then $|x|$ is just x, and so $f(|x|) = f(x) = |x-3|+1$. We understand the graph of $|x-3|+1$ the same way; if $x - 3 \geqslant 0$, i.e., if $x \geqslant 3$ then $|x-3|+1 = x-3+1 = x-2$. On the other hand if $x - 3 \leqslant 0$, i.e., if $x \leqslant 3$ then $|x-3|+1 = -(x-3)+1 = 4-x$.

That shows us that if $x \geqslant 0$ and $x \geqslant 3$ (i.e., $x \geqslant 3$) then $f(|x|) = x-2$, while if $x \geqslant 0$ and $x \leqslant 3$ (i.e., $0 \leqslant x \leqslant 3$) then $f(|x|) = 4-x$.

To finish the analysis we need to consider the other possibilities for x. If $x \leqslant 0$ then $|x| = -x$ and so $f(|x|) = f(-x) = |-x-3|+1$. Considering the two possibilities for $|-x-3|$ leads to the rest of the description of $f(|x|)$ above.

(b) The function $g(x) + g(-x)$ is just the function x^2.

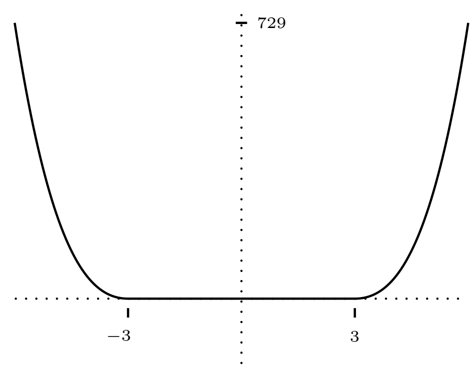


(c) The function $g\big(x^2 - 9\big)$ is equal to $\big(x^2 - 9\big)^2$ when $x^2 - 9 \geqslant 0$ and zero otherwise. The graph looks just like the graph of $(x^2 - 9)^2$ except that the portion in the interval $-3 \leqslant x \leqslant 3$ (which is the interval where $x^2 - 9 \leqslant 0$) has been cut off and replaced with the constant function 0.

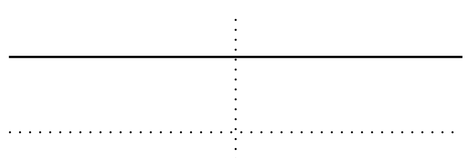
So one way of describing the function is

$$g\left(x^2 - 9\right) = \begin{cases} \left(x^2 - 9\right)^2 & \text{if } x \geqslant 3 \text{ or } x \leqslant -3 \\ 0 & \text{if } -3 \leqslant x \leqslant 3 \end{cases}$$

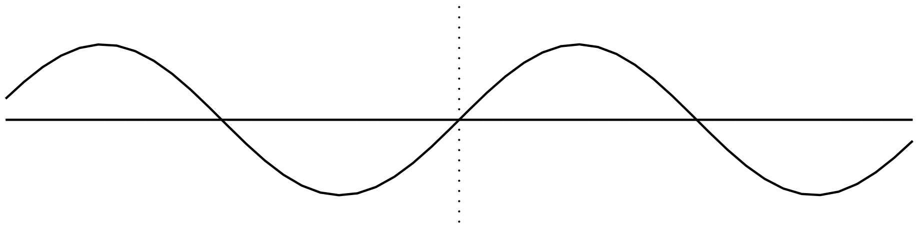
Since $(x^2-9)^2$ grows fairly rapidly, in order to get a good sketch (one which shows the essential features of the function) it is a bit better to use different scales on the x- and y-axes to be able to show what is happening.



(d) The function $h(h(x))$ is just the constant function 1; no matter what x is, $h(x)$ is either 0 or 1, both of which are rational numbers, and so h of that will be 1.



(e) If x is rational then $h(x) \cdot \sin(x)$ is just $\sin(x)$. If x is irrational, then $h(x) \cdot \sin(x)$ is zero. So the graph looks like the graph of $\sin(x)$ for the rational points and the graph of 0 for the irrational points.



One way to write the description above using the notation that we've learned is

$$h(x) \cdot \sin(x) = \begin{cases} \sin(x) & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

3. Let a, b, c, and d be real numbers, with $a < b$ and $c < d$.

(a) The definitions of $[a, b]$ and $[c, d]$ in set builder form are

$$[a, b] = \left\{ x \in \mathbb{R} \;\middle|\; a \leqslant x \leqslant b \right\}$$

and

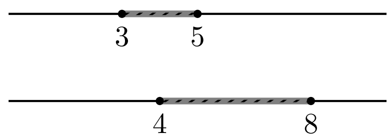
$$[c, d] = \left\{ x \in \mathbb{R} \;\middle|\; c \leqslant x \leqslant d \right\}.$$

(b) Let x be an element of $[a, b]$. Since $x \in [a, b]$, $x \leqslant b$. Since (by assumption), $b \leqslant d$, we have $x \leqslant d$.

(c) Since $b \in [a, b]$, by the assumption that $[a, b] \subseteq [c, d]$ we then know that $b \in [c, d]$. By the definition of $[c, d]$, this means that $c \leqslant b \leqslant d$. In particular, $b \leqslant d$.

Example 1 ($a = 3$, $b = 5$, $c = 4$, $d = 8$) :

(*d*) $[3, 5]$:
$[4, 8]$:



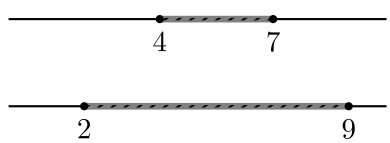
(*e*) No, Is $[3, 5]$ is not a subset of $[4, 8]$.

(*f*) The statement $c \leqslant a$ (in this case, that $4 \leqslant 3$) is not true. However, the statement that $b \leqslant d$ (in this case, that $5 \leqslant 8$) is true.

The result we proved above is that the containment holds if and only if both inequalites are true. The example above is consistent with this : we do not have containment of the intervals, and at least one of the inequalities is not true.

Example 2 ($a = 4$, $b = 7$, $c = 2$, $d = 9$) :

(*g*) $[4, 7]$:
$[2, 9]$:



(*h*) Yes, $[4, 7] \subseteq [2, 9]$.

(*i*) Both are true. The statement $c \leqslant a$ (in this case, $2 \leqslant 4$) is true, as is $b \leqslant d$ (in this case, $7 \leqslant 9$).

The example above is also consistent with the result we proved : we have containment of the intervals, and both of the inequalities are true.