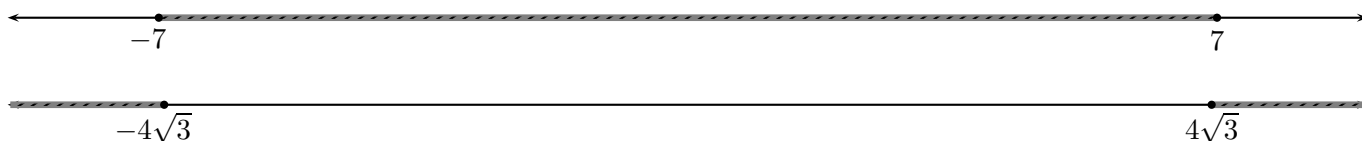


1.

- (a) We can only take the square root of non-negative numbers (numbers ≥ 0). This means that for the inner square root we're only allowed to put in x 's so that $49 - x^2 \geq 0$ or $x^2 \leq 49$. This is the same as the condition that $-7 \leq x \leq 7$.

On the other hand, to make the number we're putting in the outer square root positive, we also need to ensure that $1 - \sqrt{49 - x^2} \geq 0$, or $\sqrt{49 - x^2} \leq 1$. Squaring both sides, this is the same as $49 - x^2 \leq 1$, or $48 \leq x^2$. This happens when $x \geq 4\sqrt{3}$ or $x \leq -4\sqrt{3}$.

The domain is the set of x 's which pass both of the tests, i.e., all the x 's in $-7 \leq x \leq 7$ which satisfy either $x \geq 4\sqrt{3}$ or $x \leq -4\sqrt{3}$.



Since $4\sqrt{3} \approx 6.928203232 \dots$ there are intervals where both conditions are true (although they are very small!). The natural domain for the function is therefore those x for which $4\sqrt{3} \leq x \leq 7$ or $-7 \leq x \leq -4\sqrt{3}$, or written as a union of intervals, the set $[-7, -4\sqrt{3}] \cup [4\sqrt{3}, 7]$.

An equivalent description is that these are all the x 's satisfying the condition $4\sqrt{3} \leq |x| \leq 7$.

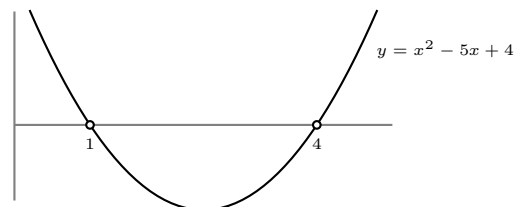
- (b) In order for $\sqrt{\ln \frac{5x-x^2}{4}}$ to make sense, we need $\ln \frac{5x-x^2}{4} \geq 0$, and therefore that $\frac{5x-x^2}{4} \geq 1$ (the condition $\ln z \geq 0$ is the same as requiring that $z \geq 1$).

The inequality $\frac{5x-x^2}{4} \geq 1$ is the same as $5x - x^2 \geq 4$, or $x^2 - 5x + 4 \leq 0$. The function $x^2 - 5x + 4$ is a parabola, opening up, with roots at $x = 1, 4$ (the latter since $x^2 - 5x + 4 = (x - 1)(x - 4)$).

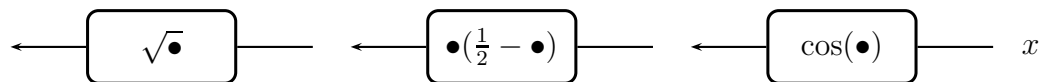
Therefore $x^2 - 5x + 4 \leq 0$ when $x \in [1, 4]$.

On the other hand, for $\frac{1}{\ln x}$ to make sense, we need $\ln x \neq 0$, or $x \neq 1$ (and $x > 0$).

Combining these two conditions, we see that the natural domain of the function is the interval $(1, 4]$.



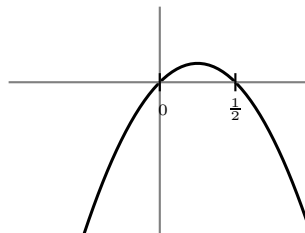
- (c) To figure out the domain of this function, it's easiest to realize that this is a composition of functions, and work through the steps of the composition. Let's use the symbol $\bullet$ to mean the input of the function, i.e., $\bullet^2$ means the function that takes its input and squares it. (We usually use x for this, but since we're going to be stringing several functions together I thought it might be too confusing to use x for all of them. We're going to be putting an x in the far end, and so what goes into the other functions won't be an x , but the result of applying the previous functions to x .) Using this weird notation, the diagram for the composition of functions is:



We want to figure out which x 's we can put in the right hand side so that every step of the composition makes sense. Let's start at the left and work backwards.

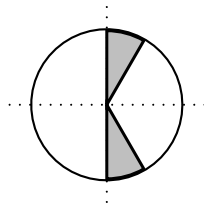
Anything we put into a square root must be positive, so we have to make sure what comes out of the middle function is positive, so the next question to ask is: what can we put into the middle function to get a positive number out?

The graph of the middle function is a parabola:



and we see that the output of the function is positive only when the input is between 0 and 0.5.

So now our question is: what x 's can we put into $\cos$ so that the output is between 0 and 0.5? On the unit circle this is just the angles so that the points have x -coordinates between 0 and 0.5:



For x 's between 0 and 2π this is the x 's satisfying $\frac{\pi}{3} \leq x \leq \frac{\pi}{2}$, or those satisfying $\frac{3\pi}{2} \leq x \leq \frac{5\pi}{3}$. That is, the union $[\frac{\pi}{3}, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, \frac{5\pi}{3}]$. In general, it is just these x 's plus some integer multiple of 2π .

Adding multiples of 2π to the first set, this is the union $\bigcup_{k \in \mathbb{Z}} [(2k + \frac{1}{3})\pi, (2k + \frac{1}{2})\pi]$.

Adding multiples of 2π to the second set, this is the union $\bigcup_{k \in \mathbb{Z}} [(2k + \frac{3}{2})\pi, (2k + \frac{5}{3})\pi]$.

To write both together, we could either take the union of those two sets :

$$\left(\bigcup_{k \in \mathbb{Z}} [(2k + \frac{1}{3})\pi, (2k + \frac{1}{2})\pi] \right) \cup \left(\bigcup_{k \in \mathbb{Z}} [(2k + \frac{3}{2})\pi, (2k + \frac{5}{3})\pi] \right)$$

or, combine the “union over all k ” parts together :

$$\bigcup_{k \in \mathbb{Z}} \left([(2k + \frac{1}{3})\pi, (2k + \frac{1}{2})\pi] \cup [(2k + \frac{3}{2})\pi, (2k + \frac{5}{3})\pi] \right).$$

NOTE: An alternate way to solve the problem “which x make $\cos(x)(\frac{1}{2} - \cos(x)) \geq 0$ ” is as follows. For the product $\cos(x)(\frac{1}{2} - \cos(x))$ to be ≥ 0 , either both factors are ≥ 0 , or both factors are ≤ 0 . Let us consider these two cases.

Case I: $\cos(x) \geq 0$ and $\frac{1}{2} - \cos(x) \geq 0$. This last inequality is the same as $\cos(x) \leq \frac{1}{2}$, and putting the inequalities together, this is the x 's so that

$$0 \leq \cos(x) \leq \frac{1}{2}.$$

The method in the first solution shows how to find these x 's.

Case II: $\cos(x) \leq 0$ and $\frac{1}{2} - \cos(x) \leq 0$. This last inequality is the same as $\frac{1}{2} \leq \cos(x)$. But, the two inequalities $\cos(x) \leq 0$ and $\cos(x) \geq \frac{1}{2}$ are incompatible — there are no such real numbers. Therefore the only possible solutions are those which come from case I.

2.

- (a) If $-A \leq z$, then multiplying by -1 reverses the inequality to give us $-z \leq A$. On the other hand, if $-z \leq A$ then multiplying by -1 gives us $-A \leq z$, so the two inequalities are equivalent.
- (b) We know (and it was written in the question) that $|z| \leq A$ is equivalent to the inequalities $-A \leq z$ and $z \leq A$. On the other hand, by part (a), the inequality $-A \leq z$ is equivalent to $-z \leq A$. Combining these, $|z| \leq A$ is equivalent to the inequalities $-z \leq A$ and $z \leq A$.

(c) By the triangle inequality we have

$$|x| = |(x + y) - y| \leq |x + y| + |y|.$$

Subtracting $|y|$ from both sides gives $|x| - |y| \leq |x + y|$.

(d) Similarly, starting with $y = (x + y) - x$ and using the triangle inequality, we get

$$|y| = |(x + y) - x| \leq |x + y| + |x|.$$

Subtracting $|x|$ from both sides gives $|y| - |x| \leq |x + y|$.

(e) By parts (c) and (d) we have

$$|x| - |y| \leq |x + y| \quad \text{and} \quad |y| - |x| \leq |x + y|.$$

Let $A = |x + y|$ and $z = |x| - |y|$. Then the above inequalities are

$$z \leq A \quad \text{and} \quad -z \leq A.$$

By part (b), this is equivalent to $|z| \leq A$. Substituting in the values of z and A we picked, this is the inequality

$$||x| - |y|| \leq |x + y|.$$

I.e., this is the lower bound part of the triangle inequality.

3. Combining the fractions over a common denominator we have $\frac{1}{a_1} + \frac{1}{a_2} = \frac{a_1 + a_2}{a_1 a_2}$, and so

$$\frac{2}{\frac{1}{a_1} + \frac{1}{a_2}} = \frac{2}{\frac{a_1 + a_2}{a_1 a_2}} = \frac{2a_1 a_2}{a_1 + a_2}.$$

The inequality we want to prove is therefore just

$$\frac{2a_1 a_2}{a_1 + a_2} \leq \sqrt{a_1 a_2}.$$

If we multiply both sides by $\frac{a_1 + a_2}{2}$, and divide both sides by $\sqrt{a_1 a_2}$ (both of which are positive so don't change the direction of the inequalities) then we get

$$\sqrt{a_1 a_2} \leq \frac{a_1 + a_2}{2},$$

i.e., we get the arithmetic-geometric mean inequality, which we know is true.

Does that mean we're done?

Well, maybe. At the very least we should write up our reasoning in a clean, clear way. There is actually a logical error (or at the very least a logical gap) in claiming that the above calculation proves the inequality we want to prove. Part of writing the argument down correctly consists in laying out the reasoning carefully, and part consists in avoiding that gap.

Here is a complete solution following the model in the homework. If you want to know what the logical gap that we're avoiding is then see the handout "A short dialogue about mathematical proofs" posted on OnQ.

We want to prove the inequality $\frac{2}{\frac{1}{a_1} + \frac{1}{a_2}} \leq \sqrt{a_1 a_2}$. Since $\frac{2}{\frac{1}{a_1} + \frac{1}{a_2}} = \frac{2a_1 a_2}{a_1 + a_2}$ this is the same as proving the inequality

$$\frac{2a_1 a_2}{a_1 + a_2} \leq \sqrt{a_1 a_2}.$$

Since a_1 and a_2 are both positive, we know that the arithmetic-geometric mean inequality

$$\sqrt{a_1 a_2} \leq \frac{a_1 + a_2}{2}$$

is true.

Multiplying both sides of the inequality above by the positive number $\sqrt{a_1 a_2}$ gives the inequality

$$a_1 a_2 \leq \left(\frac{a_1 + a_2}{2} \right) \sqrt{a_1 a_2}.$$

Multiplying both sides of this by the positive number $\frac{2}{a_1 + a_2}$ then gives the inequality

$$\frac{2a_1 a_2}{a_1 + a_2} \leq \sqrt{a_1 a_2}.$$

which is exactly what we wanted to prove. I.e., we've shown that

$$\frac{2}{\frac{1}{a_1} + \frac{1}{a_2}} \leq \sqrt{a_1 a_2}$$

for all positive real numbers a_1 and a_2 .

Another (shorter!) way of getting from the arithmetic-geometric mean inequality to the inequality we want to prove is to take the reciprocal of both sides and multiply by $a_1 a_2$. (The reciprocal is the operation $x \mapsto 1/x$. To use this argument you have to remember how the reciprocal effects inequalities, as discussed in the class of Wednesday, September 10th.)