1.

(a) Property (**1**) of the class on September 10th is that inequalities are unchanged by adding the same number to both sides. Starting with $a \leqslant A$, if we add b to both sides, we get $a + b \leqslant A + b$.

(b) Similarly, starting with the inequality $b \leqslant B$, if we add A to both sides (again using property (**1**)) we get $A + b \leqslant A + B$.

(c) The inequality from (a) is $a + b \leqslant A + b$, while the inequality from (b) is $A + b \leqslant A + B$. These two inequalities fit together :

$$a + b \leqslant A + b \leqslant A + B,$$

and show that $a + b \leqslant A + B$.

(d) The correct inequality is $ab \leqslant AB$. To prove it, we follow a plan like the one for addition. Starting with the inequality $a \leqslant A$, if we multiply by the positive number b we get $ab \leqslant Ab$ (using property (**2**) from class). Multiplying the inequality $b \leqslant B$ by the positive number A gives $Ab \leqslant AB$. Finally, these two inequalities can be combined :

$$ab \leqslant Ab \leqslant AB,$$

and show that $ab \leqslant AB$.

(e) The correct inequality is $ab \geqslant AB$. To prove it, we follow the same plan. Starting with $a \leqslant A$, multiplying by the negative number b reverses the inequality and gives $ab \geqslant Ab$. Starting with $b \leqslant B$, multipling by the negative number A gives $Ab \geqslant AB$. Putting these together we get

$$ab \geqslant Ab \geqslant AB,$$

and so $ab \geqslant AB$. (The fact that multiplying by a negative number reverses the inequality was property (**4**) from class.)

Alternate Solution. Starting with $a \leqslant A$ and $b \leqslant B$, if we multiply by -1 we get $-A \leqslant -a$ and $-B \leqslant -b$. Since all four numbers are positive, we can use the result of part (d), and conclude that $(-A)(-B) \leqslant (-a)(-b)$. Since the negative signs cancel out, this is the same as $AB \leqslant ab$.

(f) The pattern of signs where it is impossible to figure out an inequality between ab and AB (because just the initial inequalities and the signs are not enough information) is when a and b are negative, and A and B are positive.

Here are two examples to demonstrate this :

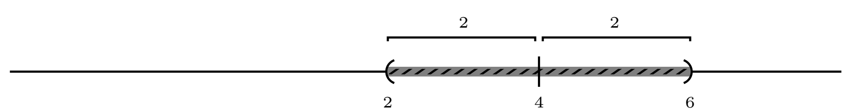
- $a = -1$, $A = 10$, $b = -2$, $B = 5$. Then we have $a \leqslant A$, and $b \leqslant B$. Since $ab = (-1)(-2) = 2$ and $AB = 10 \cdot 5 = 50$, the inequality in this case is $ab \leqslant AB$.
- $a = -10$, $A = 1$, $b = -5$, $B = 2$. Then we again have $a \leqslant A$, and $b \leqslant B$. Since $ab = (-10)(-5) = 50$ and $AB = 1 \cdot 2 = 2$, the inequality in this case is $ab \geqslant AB$.

Both these examples had the same pattern of signs, and we had $a \leqslant A$ and $b \leqslant B$ in both of them, but, as the examples show, this is not enough to determine which direction the inequality goes between ab and AB.

If we try and go through the type of arguments we used in (d) and (e), we can see the step where they fail in this case.

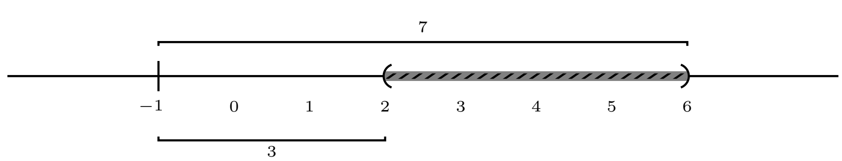
Starting with $a \leqslant A$, if we multiply by the negative number b we get $ab \geqslant Ab$. Starting with $b \leqslant B$ and multiplying by the positive number A we get $Ab \leqslant AB$. But, now we can not string the inequalities together in a row. We have $Ab \leqslant ab$ and $Ab \leqslant AB$, but this does not give us any way to compare ab and AB with each other.

2. The condition $|x - 4| < 2$ means that x must be no more than 2 away from the number 4, that is, x must be in the interval $(2, 6)$ drawn below.



(a) The number $|x + 1|$ is the distance that x is from the number -1; we want to find upper and lower bounds for this distance.

Geometric argument: If x is in the interval $(2, 6)$ the closest x can be to -1 is distance 3, and the farthest x can be from -1 is distance 7.



so the upper and lower bounds are $3 < |x+1| < 7$.

Algebraic argument: By the upper-bound triangle inequality we have

$$|x+1| = |(x-4)+5| \leqslant |x-4| + |5| < 2+5 = 7.$$

On the other hand, the lower-bound triangle inequality gives us

$$|x+1| = |(x-4)+5| \geqslant \big||x-4| - |5|\big|$$

algebraically, it's not immediately clear what to do with this inequality, so in general it's probably safer to use the picture. However, it is possible to get the lower bound by a purely algebraic argument.

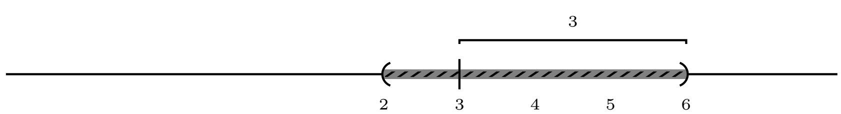
Since $|x-4| < 2$ the number $|x-4| - 5$ must be negative, and therefore

$$\big||x-4| - 5\big| = -(|x-4| - 5) = 5 - |x-4|.$$

The smallest that $5 - |x-4|$ can be occurs when the number $|x-4|$ is the largest, i.e. when $|x-4|$ is close to 2, and so $5 - |x-4| > 5 - 2 = 3$. This gives $|x+1| \leqslant 5 - |x-4| > 3$, so the upper and lower bounds are $3 < |x+1| < 7$, just like before.

(b) The number $|x-3|$ is the distance that x is from the number 3.

Geometric argument: If x is in the interval $(2,6)$ the closest x can be to 3 is distance 0 (since the number 3 is in the interval $(2,6)$), and the farthest x can be from the number 3 is distance 3.



so the upper and lower bounds are $0 \leqslant |x-3| < 3$.

Algebraic argument: By the upper-bound triangle inequality we have

$$|x-3| = |(x-4)+1| \leqslant |x-4| + |1| < 2+1 = 3.$$

The lower-bound triangle inequality gives us

$$|x-3| = |(x-4)+1| \geqslant \big||x-4| - |1|\big|$$

and again the trick is to try and figure out what the smallest number $||x-4|-1|$ can be when x is in the interval $(2, 6)$. If $x = 3$ then this is $||3-4|-1| = |1-1| = 0$ and that's the lower bound. The upper and lower bounds are therefore $0 \leqslant |x-3| < 3$ just as in the geometric argument (and like part (a), for the lower bound the geometric argument seems easer to figure out).

(c) The polynomial x^2-2x-3 factors as $(x+1)\cdot(x-3)$. Since we know that $|x+1| < 7$ (by part (a)) and that $|x-3| < 3$ (by part (b)) we can conclude that

$$|x^2 - 2x = 3| = |x+1||x-3| < 7 \cdot 3 = 21.$$

On the other hand, we also know that $|x+1| > 3$ and that $|x-3| \geqslant 0$, so we can conclude that

$$|x^2 - 2x = 3| = |x+1||x-3| \geqslant 3 \cdot 0 = 0,$$

so possible upper and lower bounds for $|x^2 - 2x - 3|$ are

$$0 \leqslant |x^2 - 2x - 3| < 21.$$

(d) We know that $3 < |x+1| < 7$. Since all the numbers are positive, taking the reciprocal reverses the inequalities:

$$\frac{1}{3} > \frac{1}{|x+1|} > \frac{1}{7}.$$

Now we repeat the same kind of argument as in part (c). Since $|x-3| < 3$ and $\dfrac{1}{|x+1|} < \dfrac{1}{3}$ we have

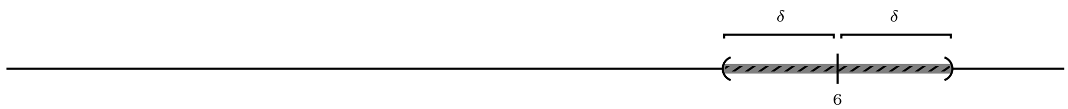
$$\frac{|x-3|}{|x+1|} = |x-3| \cdot \frac{1}{|x+1|} < 3 \cdot \frac{1}{3} = 1,$$

and similarly since $0 \leqslant |x-3|$ and $\frac{1}{7} < \dfrac{1}{|x+1|}$ we have

$$\frac{|x-3|}{|x+1|} = |x-3| \cdot \frac{1}{|x+1|} \geqslant 0 \cdot \frac{1}{7} = 0.$$

So the upper and lower bounds are $0 \leqslant \dfrac{|x-3|}{|x+1|} < 1$.

3. Suppose we know that $|x-6| < \delta$ where δ is some positive number. That means that we know that x is in the interval $(6-\delta, 6+\delta)$ shown below:

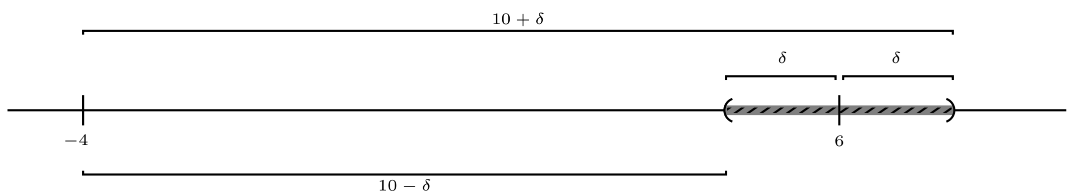


(a) We want to find an upper bound for

$$\left|\frac{5x}{x+4} - 3\right| = \left|\frac{5x - 3(x+4)}{x+4}\right| = \left|\frac{2x-12}{x+4}\right| = \left|\frac{2(x-6)}{x+4}\right| = 2 \cdot |x-6| \cdot \frac{1}{|x+4|}.$$

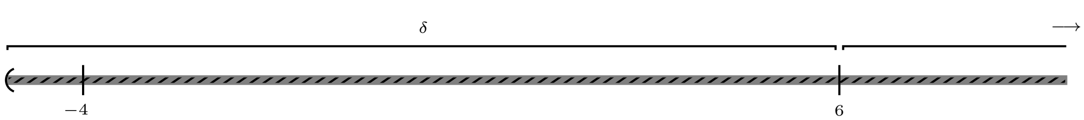
We already know that $|x-6| < \delta$, so we just have to understand the size of $\frac{1}{|x+4|}$. We do this by using the same method as in question 2; we first find bounds for $|x+4|$ and then take the reciprocal.

The number $|x+4|$ is the distance that x is from the number -4. Drawing the picture



it seems that if x is in the interval $(6-\delta, 6+\delta)$ then the farthest that x can be from -4 is distance $10+\delta$, and the closest that x can be to -4 is distance $10-\delta$. That is, from the picture it looks like $10 - \delta < |x+4| < 10 + \delta$.

That's true, but we do have to be a little careful – if δ is big enough (i.e., $\delta > 10$) then -4 will actually be in the interval $(6-\delta, 6+\delta)$:



and then the closest that x can be to -4 is distance 0. A lower bound like $0 \leqslant |x+4|$ will not help us, since when we take the reciprocal all that tells us is that $\frac{1}{|x+4|}$ is less than ∞. This is true, but not useful!

Let us assume that $\delta < 10$. Then we can say that $10 - \delta < |x+4|$, and that $10-\delta$ is positive. Taking the reciprocal then gives $\frac{1}{|x+4|} < \frac{1}{10-\delta}$.

We could also follow the suggestion of the homework and assume that $\delta \leqslant 9$. Then we have $10-9 \leqslant 10-\delta$, and so $1 < |x+4|$, and taking the reciprocal gives $\frac{1}{|x+4|} < 1$.

Using the first estimate (that $\frac{1}{|x+4|} < \frac{1}{10-\delta}$) we get

$$\left|\frac{5x}{x+4} - 3\right| = 2 \cdot |x-6| \cdot \frac{1}{|x+4|} < 2 \cdot \delta \cdot \frac{1}{10-\delta} = \frac{2\delta}{10-\delta}.$$

While using the second estimate (that $\frac{1}{|x+4|} < 1$) gives

$$\left|\frac{5x}{x+4} - 3\right| = 2 \cdot |x-6| \cdot \frac{1}{|x+4|} < 2 \cdot \delta \cdot = 2\delta.$$

(*b*) If $|x-6| < 4$ (i.e., $|x-6| < \delta$ with $\delta = 4$) then the first upper bound from part (a) gives us

$$\left|\frac{5x}{x+4} - 3\right| < \frac{2\delta}{10-\delta} \leqslant \frac{2 \cdot 4}{10-4} = \frac{8}{6} = \frac{4}{3}.$$

(*c*) We want to find δ so that $\frac{2\delta}{10-\delta} \leqslant \frac{1}{2}$. Since $10-\delta$ is positive the inequality will stay in the same direction if we multiply by $10-\delta$. Multiplying both sides of the inequality by $10-\delta$ and then by 2 gives the inequality $4\delta \leqslant 10-\delta$, or $5\delta \leqslant 10$, or $\delta \leqslant 2$. So, any positive δ with $\delta \leqslant 2$ will do.

NOTE: The question only asked to find a single value of δ that would make $\left|\frac{5x}{x+4} - 3\right| < \frac{1}{2}$, so a single number (e.g., "$\delta = 1$") would be a sufficient answer. The point of the calculation above was to show that it is straightforward to find the general answer.

(*d*) There are many different ways to answer this question. Two possible solutions are given below.

SOLUTION I: We want to find δ so that $\frac{2\delta}{10-\delta} \leqslant \frac{1}{m}$. Multiplying both sides by m gives $\frac{2m\delta}{10-\delta} \leqslant 1$. Since $10-\delta$ is positive (we're assuming that $\delta < 9$) we can multiply by $10-\delta$ to get $2m\delta \leqslant 10-\delta$, or $(2m+1)\delta \leqslant 10$ which has solution $\delta \leqslant \frac{10}{2m+1}$.

So, one solution is to simply take $\delta \leqslant \dfrac{10}{2m+1}$.

SOLUTION II: If we use instead the second upper bound from part (a) (the one that assumes that $\delta \leqslant 9$), the problem is then to find δ so that $2\delta \leqslant \frac{1}{m}$. That is an easy problem to solve : we can take $\delta = \min(9, \frac{1}{2m})$, or even just $\delta = \frac{1}{2m}$, since we know that m is a positive integer, so $\frac{1}{2m}$ will always be smaller than 9.

Thus, a second possible answer is to take $\delta = \frac{1}{2m}$.