1.

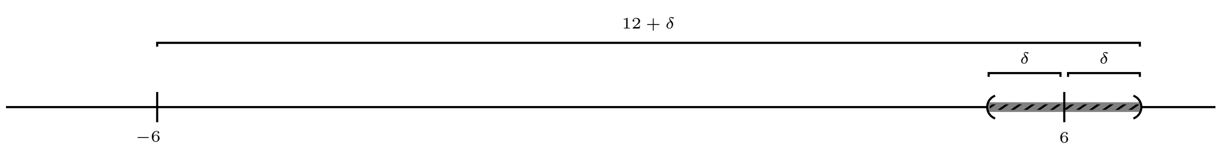
(a) $\lim_{x \to 6} (x^2 + 4) = 40.$

INITIAL INVESTIGATION : We want to understand

$$|(x^2 + 4) - 40| = |x^2 - 36| = |(x - 6)(x + 6)| = |x - 6| \cdot |x + 6|.$$

Specifically, we want to understand how the condition $|x - 6| < \delta$ effects $|x - 6| \cdot |x + 6|$ so that we can make it less than ϵ.

Here is the picture :



The condition $|x - 6| < \delta$ means that x must be within δ of 6, and for those x's, we want to know an upper bound for $|x + 6|$, the distance from x to -6. From the picture, this distance is no more than $12 + \delta$, and so we get the estimate $|x+6| < 12+\delta$. Alternatively this follows immediately from the triangle inequality:

$$|x + 6| = |(x - 6) + 12| \leqslant |x - 6| + |12| < \delta + 12.$$

Going back to the original function this means that if $|x - 6| < \delta$ then

$$|(x^2 + 4) - 40| = |x - 6| \cdot |x + 6| < \delta(\delta + 12).$$

We now want to understand how to pick δ so that this is less than ϵ (after someone gives us an ϵ). The factor $\delta + 12$ is slightly awkward to deal with, and so let's use our freedom to pick δ to allow us to estimate it by something simpler. Let's suppose that $\delta < 1$ (any fixed positive number would work here). Then $12+\delta < 13$ and so $\delta(12 + \delta) < \delta \cdot 13 = 13\delta$. Now it's easy to solve $13\delta \leqslant \epsilon$, we just pick any $\delta < \frac{\epsilon}{13}$. So our argument works whenever δ is both less than 1 and less than $\frac{\epsilon}{13}$, and so to make sure that both of these conditions are satisfied we pick δ less then the minimum of these two numbers.

Now we are ready to write the ϵ-δ proof.

Claim: $\lim_{x \to 6} (x^2 + 4) = 40$

Proof: Suppose that we're given $\epsilon > 0$. Pick any $\delta < \min(1, \frac{\epsilon}{13})$. Then

$$|(x+4) - 40| = |x^2 - 36| = |x-6| \cdot |x+6| < \delta(\delta+12) < 13\delta < \epsilon.$$

Since this argument works for any $\epsilon > 0$, we have $\lim\limits_{x \to 6}(x^2+4) = 40$ by the definition of limit. $\square$

(*b*) $\lim\limits_{x \to -4}(3x^2 + 3x + 4) = 40.$

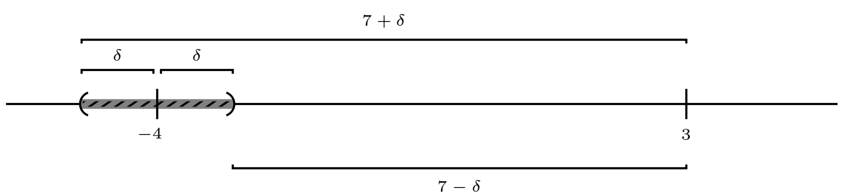
Initial Investigation : We want to understand

$$|(3x^2+3x+4)-40| = |3x^2+3x-36| = 3|x^2+x-12| = 3|(x-3)(x+4)| = 3|x-3|\cdot|x+4|.$$

Since the limit is as x goes to -4, we control the size of $|x-(-4)| = |x+4|$ by controlling the size of δ. We next need to find an upper bound for $|x-3|$ when $0 < |x+4| < \delta$. By the upper bound triangle inequality, when $|x+4| < \delta$,

$$|x-3| = |(x+4) - 7| \leqslant |x+4| + |7| < \delta + 7.$$

Alternatively, we can also get this by looking at the picture :



When $|x+4| < \delta$ (i.e., when x is in the interval $(-4-\delta, -4+\delta)$) we see that $|x-3|$ can be at most $7+\delta$.

Going back to the original function, this means that if $|x+4| < \delta$, then

$$|(3x^2+3x+4) - 40| = 3|x-3| \cdot |x+4| < 3(7+\delta)\delta.$$

We next want to understand, given an $\epsilon > 0$, how to pick δ so that $3(7+\delta)\delta \leqslant \epsilon$. Let us again make an assumption on δ to make this easier. Let's assume that $\delta \leqslant 3$ (or $\delta \leqslant 2$, or $\delta \leqslant 5$,...any positive upper bound will do). If $\delta \leqslant 3$, then $7+\delta \leqslant 10$, and then we get the estimate

$$(3x^3+3x+4) - 40)| \leqslant 3(7+\delta)\delta \leqslant 3 \cdot 10 \cdot \delta = 30\delta.$$

Given ϵ, in order to make $30\delta \leqslant \epsilon$, one choice is $\delta = \frac{\epsilon}{30}$. More precisely, in order to ensure that $\delta \leqslant \frac{\epsilon}{30}$, and that our assumption $\delta \leqslant 3$ is true, we should pick $\delta \leqslant \min(\frac{\epsilon}{30}, 3)$. Now we are ready to write the proof.

Claim: $\lim_{x\to -4}(3x^2+3x+4) = 40$.

Proof. Given $\epsilon > 0$ pick $\delta = \min(\frac{\epsilon}{30}, 3)$. Then, if $0 < |x+4| < \delta$,

$$|x-3| = |(x+4)-7| \leqslant |x+4| + |7| < \delta + 7 \leqslant 10,$$

and therefore

$$|(3x^2+3x+4)-40| = |3x^2+3x-36| = 3|x-3|\cdot|x+4| \leqslant 3\cdot 10\cdot\delta \leqslant 30\cdot\frac{\epsilon}{30} = \epsilon.$$

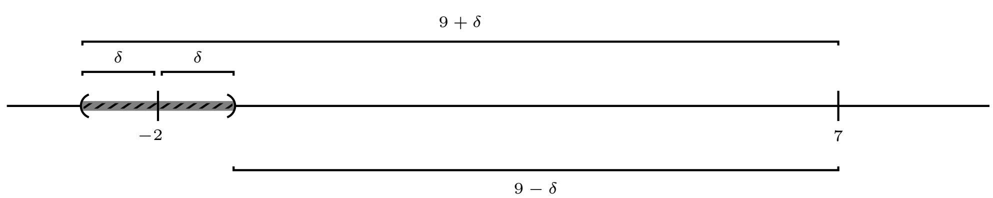
Therefore, by the definition of limit, $\lim_{x\to -4}(3x^2+3x+4) = 40$. □

(c) $\lim_{x\to -2}\dfrac{4x-19}{x-7} = 3$.

INITIAL INVESTIGATION : We want to understand

$$\left|\frac{4x-19}{x-7} - 3\right| = \left|\frac{(4x-19)-3(x-7)}{x-7}\right| = \left|\frac{x+2}{x-7}\right| = \frac{|x+2|}{|x-7|}.$$

Because the limit is as x is going to -2, we control the size of $|x-(-2)| = |x+2|$, by controlling the size of δ. To find an upper bound on $\frac{1}{|x-7|}$ we find a positive lower bound for $|x-7|$ and take the reciprocal. If $|x+2| < \delta$, we have the following picture :



As long as δ is small enough so that the interval $(-2-\delta, -2+\delta)$ does not contain 7 (i.e., as long as $\delta < 9$), then an x in the interval is at least $9-\delta$ away from 7, and we get the lower bound $|x-7| \geqslant 9-\delta$. To ensure that $\delta < 9$, let us make an assumpton δ, that $\delta \leqslant 7$. (Replacing 7 by any other positive number less than 9 would also work.) Then $|x-7| \geqslant 9-\delta \geqslant 9-7 = 2$.

We can also get this from the lower bound triangle inequality. Assuming that $|x+2| < \delta$,

$$|x-7| = |(x+2)-9| \geqslant ||9| - |x+2|| \geqslant 9-\delta.$$

If $\delta \leqslant 7$, then $9-\delta \geqslant 2$, and we again get $|x-7| \geqslant 2$. Taking the reciprocal, we get the upper bound $\frac{1}{|x-7|} \leqslant \frac{1}{2}$.

Returning to our original problem, we now know that if $\delta \leqslant 7$ and $0 < |x+2| < \delta$, then

$$\left|\frac{4x-19}{x-7} - 3\right| = \frac{|x+2|}{|x-7|} \leqslant \delta \cdot \frac{1}{2} = \frac{\delta}{2}.$$

Given $\epsilon > 0$, one choice of δ so that $\frac{\delta}{2} \leqslant \epsilon$ is to choose $\delta \leqslant 2\epsilon$. In order to also ensure that we have $\delta \leqslant 7$, we should choose $\delta \leqslant \min(2\epsilon, 7)$.

Now we are ready to write the ϵ-δ proof.

Claim. $\displaystyle\lim_{x\to -2} \frac{4x-19}{x-7} = 3.$

Proof. Given $\epsilon > 0$ set $\delta = \min(2\epsilon, 7)$. Then, if $0 < |x+2| < \delta$, we have

$$|x-7| = |(x+2) - 9| \geqslant ||9| - |x+2|| \geqslant 9 - \delta \geqslant 9 - 7 = 2,$$

where in the last step we have used the fact that $\delta \leqslant 7$. Taking the reciprocal, this means that $\frac{1}{|x-7|} \leqslant \frac{1}{2}$. Therefore, when $0 < |x+2| < \delta$,

$$\left|\frac{4x-19}{x-7} - 3\right| = \left|\frac{x+2}{x-7}\right| = |x+2| \cdot \frac{1}{|x-7|} \leqslant 2\epsilon \cdot \frac{1}{2} = \epsilon.$$

Therefore, by the definition of the limit, $\displaystyle\lim_{x\to -2} \frac{4x-19}{x-7} = 3.$

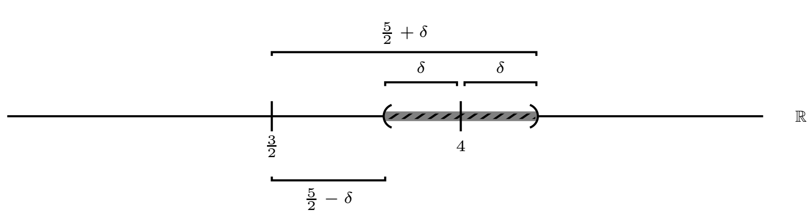
(*d*) $\displaystyle\lim_{x\to 4} \frac{3x-2}{2x-3} = 2.$

Initial Investigation : We want to understand

$$\left|\frac{3x-2}{2x-3} - 2\right| = \left|\frac{3x-2-2(2x-3)}{2x-3}\right| = \left|\frac{-x+4}{2x-3}\right| = |x-4| \cdot \frac{1}{2} \cdot \frac{1}{|x-\frac{3}{2}|}.$$

We want to understand how the condition $|x-4| < \delta$ effects the expression above so that we can see how to pick δ to make the expression less than ϵ. The only mysterious term is $\frac{1}{|x-\frac{3}{2}|}$; to understand this we start by finding bounds for $|x-\frac{3}{2}|$.

By drawing the picture:



we see that if $|x-4| < \delta$ then $\frac{5}{2} - \delta < |x-\frac{3}{2}|$, and that $|x-\frac{3}{2}| < \frac{5}{2} + \delta$.

(NOTE: If $\delta > \frac{5}{2}$ then the lower bound above is *negative*, and in fact the interval $|x-4| < \delta$ would contain $\frac{3}{2}$ so that a better lower bound would be zero. This is the same thing that happened in question 3(a) of Homework 3.)

We can also get the upper bound by using the triangle inequality:

$$\left|x - \frac{3}{2}\right| = \left|(x-4) + \frac{5}{2}\right| \leqslant |x-4| + \left|\frac{5}{2}\right| \leqslant \delta + \frac{5}{2}.$$

For the lower bound it is safer just to use the picture.

In order to get an upper bound for $\frac{1}{|x-\frac{3}{2}|}$ we want to take the lower bound for $|x - \frac{3}{2}|$ and take the reciprocal. In order to be sure that the inequality switches direction, we need to know that $\frac{5}{2} - \delta$ is positive; if $\frac{5}{2} - \delta$ is negative the inequality will still go in the same direction. Since we get to control δ, we can assume that this is true.

Assume that $\delta < \frac{5}{2}$. Then since $\frac{5}{2} - \delta$ is positive, and since $\frac{5}{2} - \delta < |x - \frac{3}{2}|$ when we take the reciprocal we get

$$\frac{1}{|x-\frac{3}{2}|} < \frac{1}{\frac{5}{2} - \delta}.$$

Putting these steps together we see that if $\delta < \frac{5}{2}$ and if $|x-4| < \delta$, then

$$\left|\frac{3x-2}{2x-3} - 2\right| = |x-4| \cdot \frac{1}{2} \cdot \frac{1}{|x-\frac{3}{2}|}. < \delta \cdot \frac{1}{2} \cdot \frac{1}{\frac{5}{2} - \delta} = \frac{\delta}{5-2\delta}.$$

We now want to understand how to pick δ so that $\frac{\delta}{5-2\delta} < \epsilon$ (after someone gives us an ϵ). This would be easier if the denominator were simpler, and perhaps we can make another assumption about δ that will let us replace the denominator with something easier to handle.

The simplest assumption is to assume that $\delta < 2$, then $5 - 2\delta > 1$, and so $\frac{1}{5-2\delta} < 1$. Putting these last two steps together, we see that if $\delta < 2$ (and so $\delta < \frac{5}{2}$ too) and if $|x-4| < \delta$, then

$$\left|\frac{3x-2}{2x-3} - 2\right| < \delta \cdot \frac{1}{5-2\delta} < \delta \cdot 1 = \delta.$$

Now if we're given an ϵ, it's easy to find a δ such that $\delta \leqslant \epsilon$ (we just pick any δ less than ϵ!). Now we just have to put these steps together to show that the limit is 2. Since we used the assumption that $\delta < 2$ (and $\delta < \frac{5}{2}$) in our arguments above, we'll need to ensure that any δ we pick still satisfies this condition, as well as the condition that $\delta \leqslant \epsilon$. The way that we make sure that both of them are satisfied is by picking δ less than the minimum of these two numbers.

Now we are ready to write the ϵ-δ proof.

Claim: $\lim_{x\to 4} \dfrac{3x-2}{2x-3} = 2$.

Proof: Suppose that we're given $\epsilon > 0$. Pick any $\delta < \min(2, \epsilon)$. Then

$$\left|\frac{3x-2}{2x-3} - 2\right| = |x-4| \cdot \frac{1}{2} \cdot \frac{1}{|x - \frac{3}{2}|} < \delta \cdot \frac{1}{5-2\delta} < \delta \cdot 1 = \delta < \epsilon.$$

Since this argument works for any $\epsilon > 0$, by the definition of the limit we have $\lim_{x\to 4} \dfrac{3x-2}{2x-3} = 2$. $\square$

NOTE: It is possible to solve the inequality $\frac{\delta}{5-2\delta} < \epsilon$ directly and so avoid the step where we assume that $\delta < 2$. Multiplying both sides of the inequality by $5 - 2\delta$ (which is positive, since we're assuming that $\delta < \frac{5}{2}$) we get $\delta < \epsilon(5 - 2\delta)$. Multiplying out and rearranging gives $(2\epsilon + 1)\delta < 5\epsilon$ and finally dividing gives $\delta < \frac{5\epsilon}{2\epsilon+1}$.

An alternate solution using this method would therefore start by picking $\delta < \min(\frac{5}{2}, \frac{5\epsilon}{2\epsilon+1})$ and then show how this leads to $|\frac{3x-2}{2x-3} - 2| < \epsilon$.

2.

(a) By the triangle inequality,

$$|x^2 + cx + c^2| \leqslant |x^2| + |cx| + |c^2| = |x|^2 + |c||x| + |c|^2,$$

where in the last equality we have used the identities $|cx| = |c||x|$ and $|c^2| = |c|^2$.

(b) This again follows from the triangle inequality; if $|x - c| < 1$ then

$$|x| = |(x-c) + c) \leqslant |x - c| + |c| < 1 + |c|.$$

One can also use the lower bound triangle inequality :

$$1 > |x - c| \geqslant \big||x| - |c|\big| \geqslant |x| - |c|,$$

which we can rearrange to get $|x| \leqslant 1 + |c|$.

Finally, a third way to see this inequality is by drawing a picture, although to draw an accurate picture we have to consider the cases that c is positive or negative (and perhaps even if $|c| < 1$ or not).

(c) By part (a) we have $|x^2+cx+c^2| \leqslant |x|^2+|c||x|+|c|^2$, and by part (b) we have $|x| < (1+|c|)$ if $|x-c|<1$. Combining them, we see that if $|x-c|<1$ then

$$|x^2+cx+c^2| \leqslant |x|^2+|c||x|+|c|^2 < (1+|c|)^2+|c|(1+|c|)+|c|^2.$$

(d) Suppose that ϵ is a positive number, and that $|x-c| < \min\left(1, \frac{\epsilon}{(1+|c|)^2+|c|(1+|c|)+|c|^2}\right)$.

This means that we can assume that $\delta < 1$ (and so $|x| < (1+|c|)$) and also that $\delta < \frac{\epsilon}{(1+|c|)^2+|c|(1+|c|)+|c|^2}$. If $|x-c|<\delta$ then

$$\begin{aligned} |x^3-c^3| &= |x-c||x^2+cx+c^2| < \delta\left((1+|c|)^2+|c|(1+|c|)+|c|^2\right) \\ &< \left(\frac{\epsilon}{(1+|c|)^2+|c|(1+|c|)+|c|^2}\right)\cdot\left((1+|c|)^2+|c|(1+|c|)+|c|^2\right) \\ &= \epsilon. \end{aligned}$$

(e) *Claim:* $\lim\limits_{x\to c} x^3 = c^3$.

Proof: Suppose that we're given $\epsilon > 0$. Pick $\delta < \min\left(1, \frac{\epsilon}{(1+|c|)^2+|c|(1+|c|)+|c|^2}\right)$. Then if $|x-c|<\delta$, $|x^3-c^3|<\epsilon$ by part (d). Since this argument works for any $\epsilon>0$, by the definition of the limit we have $\lim\limits_{x\to c} x^3 = c^3$. □

3. Squeeze!

(a) Since $-1 \leqslant \sin(1/x) \leqslant 1$ if we multiply by x^4+x^2 (which is positive when $x \neq 0$) we have $-(x^4+x^2) \leqslant (x^4+x^2)\sin(1/x) \leqslant (x^4+x^2)$ for all $x \neq 0$.

Since $\lim\limits_{x\to 0} -(x^4+x^2) = -(0^4+0^2) = 0 = \lim\limits_{x\to 0} x^4+x^2$, the squeeze theorem tells us that $\lim\limits_{x\to 0}(x^4+x^2)\sin(1/x)$ exists and is equal to 0.

(b) Since $\sin(1/x)$ is always between -1 and 1, and since $\cos(x)$ is always between -1 and 1, their product is also always between -1 and 1, in other words we always have

$$-1 \leqslant \cos(x)\sin(1/x) \leqslant 1.$$

Here is another way to arrive at this conclusion (one which may seem safer): Saying that $-1 \leqslant \cos(x) \leqslant 1$ is the same as saying that $|\cos(x)| \leqslant 1$. Similarly saying that $-1 \leqslant \sin(\frac{1}{x}) \leqslant 1$ is the same as saying that $|\sin(\frac{1}{x})| \leqslant 1$. Everything in these last two inequalities is non-negative, so we can multiply them to get

$$|\cos(x)\sin(\tfrac{1}{x})| = |\cos(x)|\cdot|\sin(\tfrac{1}{x})| \leqslant 1\cdot 1 = 1.$$

Finally, we reverse the previous argument and note that the inequality $|\cos(x)\sin(\frac{1}{x})| \leqslant 1$ is equivalent to $-1 \leqslant \cos(x)\sin(\frac{1}{x}) \leqslant 1$.

Multiplying this by the positive number x^2 we get

$$-x^2 \leqslant x^2\cos(x)\sin(1/x) \leqslant x^2,$$

and adding 8 gives

$$8 - x^2 \leqslant 8 + x^2\cos(x)\sin(1/x) \leqslant 8 + x^2.$$

Since $\lim_{x\to 0} 8 - x^2 = 8 = \lim_{x\to 0} 8 + x^2$ the squeeze theorem tells us that

$$\lim_{x\to 0} 8 + x^2\cos(x)\sin(1/x)$$

exists and is equal to 8.