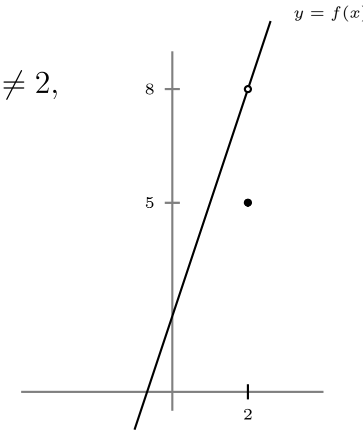
1.



(a) A graph of the function f is shown at right. For every $x \neq 2$, $f(x) = 3x + 2$, while when $x = 2$, $f(2) = 5$.

(b) Yes, $\lim_{x\to 2} f(x)$ exists, and is equal to 8.

(c) <u>Initial Investigation</u> : Given $\epsilon > 0$, we want to find $\delta > 0$ so that if $0 < |x-2| < \delta$ then $|f(x) - 8| \leqslant \epsilon$.

But, the condition $0 < |x-2|$ means that x is not allowed to be 2 (as in all cases with limits, we don't allow x to be equal to the limit point). And, if $x \neq 2$, then $f(x) = 3x + 2$. So, as long as $0 < |x - 2| < \delta$, we have $f(x) = 3x + 2$, and so $|f(x) - 8| = |(3x + 2) - 8| = |3x - 6| = 2|x - 3| \leqslant 3\delta$. In order to ensure that this is $\leqslant \epsilon$, we can take $\delta = \frac{\epsilon}{3}$.

Proof. Given $\epsilon > 0$, set $\delta = \frac{\epsilon}{3}$. Then, if $0 < |x - 2| < \delta$, $f(x) = 3x + 2$, and so $|f(x) - 8| = |(3x + 2) - 8| = 3|x - 2| \leqslant 3\delta = 3 \cdot \frac{\epsilon}{3} = \epsilon$. Therefore $\lim\limits_{x\to 2} f(x) = 8$.

(d) f is *not* continuous at $x = 2$. For f to be continuous at $x = 2$ means that $\lim\limits_{x\to 2} f(x) = f(2)$. As we know, this short equation really encodes three conditions :

- (i) $\lim\limits_{x\to 2} f(x)$ exists.
- (ii) f is defined at $x = 2$
- (iii) The number $\lim\limits_{x\to 2} f(x)$ is equal to the number $f(2)$.

In our case, both (i) and (ii) are satisfied. We have seen above that $\lim_{x\to 2} f(x)$ exists, and f is defined at $x = 2$. However, the value of the limit is 8, and $f(2) = 5$, so condition (iii) is not satisfied.

2. Here is the table.

(*a*)

x	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5
$g(x)$	2	1	0	0	0	1	2

To see that this these are the right values, it is helpful to write down the tables for $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ separately

x	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5
$\lceil x \rceil$	-1	-1	0	0	1	1	2
$\lfloor x \rfloor$	-2	-1	-1	0	0	1	1

.

Then the value of $g(x)$ is the product of the entries in each column of the table above. E.g., $g(-1.5) = (-1)(-2) = 2$, $g(-1) = (-1)(-1) = 1$, $g(-0.5) = (0)(-1) = 0$, etc.

(*b*) Yes, $\lim_{x\to 3^+} g(x)$ exists, and has value 12.

To see why, $\lim_{x\to 3^+} g(x)$ is asking what happens to $g(x)$ as x approaches 3 from above (that is, from numbers larger than 3). For all $x \in (3, 4)$, i.e., all x larger than 3, but less than 4, $\lceil x \rceil = 4$ and $\lfloor x \rfloor = 3$, so that $g(x) = 4 \cdot 3 = 12$.

Thus, once x is close enough to 3 from above (specifically, once $3 < x < 4$, or $0 < x - 3 < 1$) then $g(x) = 12$, and so $\lim_{x\to 3^+} g(x) = 12$.

(*c*) No, $g(x)$ is not continuous at $x = 3$. The reason is that $\lim_{x\to 3} g(x)$ does not exist.

Arguing as in part (*b*), we have $\lim_{x\to 3^-} g(x) = 6$, since, when $x \in (2, 3)$ we have $\lceil x \rceil = 3$ and $\lfloor x \rfloor = 2$, so that $g(x) = 3 \cdot 2 = 6$. As in part (*b*), this implies that $\lim_{x\to 3^-} g(x) = 6$.

We know that $\lim_{x\to 3} g(x)$ exists if and only if $\lim_{x\to 3^+} g(x)$ and $\lim_{x\to 3^-} g(x)$ both exist and have the same value. We have seen above that both of these one-sided limits do exist, but have different values. Therefore $\lim_{x\to 3} g(x)$ does not exist.

For g to be continuous at $x = 3$ means that $\lim_{x\to 3} g(x) = g(3)$. As before, this is really three conditions

(1) $\lim_{x\to 3} g(x)$ exists.

(2) g is defined at $x = 3$

(3) The limit $\lim_{x\to 3} g(x)$ is equal to the number $g(3)$.

As we have seen, condition (1) is not satisfied. Condition (2) is satisfied, g is defined at $x = 3$ and has value $g(3) = 9$. Given that the limit $\lim_{x\to 3} g(x)$ doesn't exist, it does not make sense to ask if this non-existent limit is equal to $g(3) = 9$, i.e., to ask about condition (3).

(d) g is continuous at $x = 0$.

For $x \in (-1, 0]$, $\lceil x \rceil = 0$, and so for $x \in (-1, 0]$, $g(x) = 0 \cdot \lfloor x \rfloor = 0$.

Similarly, for $x \in [0, 1)$, $\lfloor x \rfloor = 0$, and so for those x, $g(x) = \lceil x \rceil \cdot 0 = 0$.

Combining these, for all $x \in (-1, 1)$, $g(x) = 0$. Therefore,

(1) $\lim_{x \to 0} g(x)$ exists, and is equal to 0.

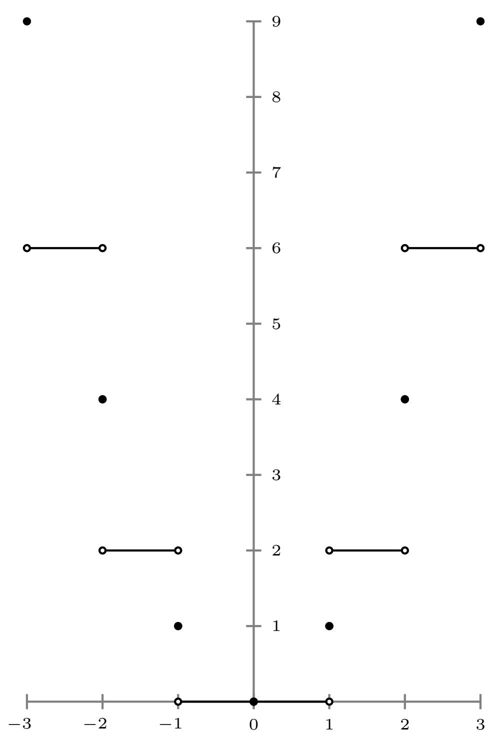
(2) g is defined at $x = 0$, with value $g(0) = 0$.

(3) The limit $\lim_{x \to 0} g(x)$ (namely 0) is equal to the number $g(0)$.

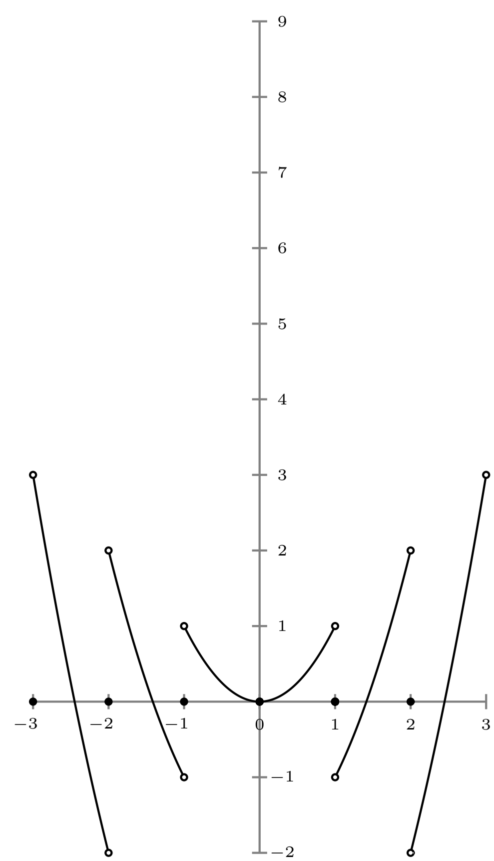
Thus, $\lim_{x \to 0} g(x) = g(0)$, and so g is continuous at $x = 0$.

Below is a graph of $g(x)$ on $[-3, 3]$, along with a graph of $x^2 - g(x)$, that is, the graph of $x^2 - \lceil x \rceil \cdot \lfloor x \rfloor$. (This second graph is included just because I find it interesting. It was not mentioned in the original problem.)

$y = g(x)$



$y = x^2 - g(x)$



3.

(a) The unique value of a so that the limit $\lim_{x\to 2} h(x)$ exists is $a = 6$.

If $a = 6$, then when $x \neq 2$ we have

$$h(x) = \frac{x^2+6x-16}{2x-4} = \frac{(x-2)(x+8)}{2(x-2)} = \tfrac{1}{2}(x+8),$$

and therefore

$$h(x) = \begin{cases} \frac{1}{2}(x+8) & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$$

So that

$$\lim_{x\to 2} h(x) \overset{(1)}{=} \lim_{x\to 2} \tfrac{1}{2}(x+8) \overset{(2)}{=} \tfrac{1}{2}(2+8) = 5,$$

with reasons

(1) The limit is as $x \to 2$, and when $x \neq 2$, $h(x) = \frac{1}{2}(x+8)$,

(2) $\frac{1}{2}(x+8)$ is a polynomial, and so continuous at $x = 2$ (and in fact, continuous on all of $\mathbb{R}$), and therefore the limit as $x \to 2$ is the value of the polynomial at $x = 2$.

The above argument shows that there is a limit when $a = 6$, and no further explanation was required in the question. But, how could we have found this value of a? I.e., what ideas would lead us to figure out that $a = 6$ was something to try?

Here is one such way of thinking.

Except for at $x = 2$, the function $h(x)$ is given by a rational function $h(x) = \frac{x^2+ax-16}{2x-4}$. Let $p(x) = x^2 + ax - 16$ be the numerator and $q(x) = 2x - 4$ be the denominator.

We first note that as $x \to 2$, $q(x)$ goes to 0. (Since $q(x)$ is a polynomial, and therefore a continuous function, we have $\lim_{x\to 2} q(x) = q(2) = 2 \cdot 2 - 4 = 0$.)

Claim : In order for $\lim_{x\to 2} h(x)$ to exist, we must also have $\lim_{x\to 2} p(x) = 0$, and therefore (since $\lim_{x\to 2} p(x) = p(2)$), $p(2) = 0$.

Proof. Assume that $\lim_{x\to 2} h(x)$ exists, and let L be the limit. Then, by the theorem on limits and products,

$$\lim_{x\to 2} p(x) = \lim_{x\to 2} \frac{p(x)}{q(x)} \cdot q(x) = \left(\lim_{x\to 2} h(x)\right) \cdot \left(\lim_{x\to 2} q(x)\right) = L \cdot 0 = 0. \qquad \square$$

One can also prove the claim by reasoning the other way: We know that $\lim_{x\to 2} p(x)$ exists. (Since p is a polynomial, and continuous, the limit exists and is equal to

$p(2)$.) Suppose that $\lim_{x\to 2} p(x)$ is not zero. Then, as $x \to 2$, the numerator of $h(x)$ is going to some nonzero number, while the denominator is going to 0. The ratio must be going to $\pm\infty$!

So, again, if $\lim_{x\to 2} h(x)$ exists we must have $p(2) = 0$. Plugging 2 into p we get

$$0 = p(2) = (2)^2 + a(2) - 16 = 4 + 2a - 16 = 2a - 12 = 2(a - 6),$$

and for this to be true we must have $a = 6$.

(b) When $a = 6$, the function $h(x)$ is continuous at $x = 2$.

As shown in part (a), when $a = 6$, $\lim_{x\to 2} h(x) = 5$, and since $h(2) = 5$, this means we have $\lim_{x\to 2} h(x) = h(2)$, which is the condition for h to be continuous at $x = 2$.

4. $\displaystyle\lim_{x\to\infty} \frac{\sin(x)}{x^2} = 0$.

Proof. For all x we have $-1 \leqslant \sin(x) \leqslant 1$. When $x \neq 0$, $\frac{1}{x^2} \geqslant 0$, and multiplying the previous inequality by $\frac{1}{x^2}$ gives

$$-\frac{1}{x^2} \leqslant \frac{\sin(x)}{x^2} \leqslant \frac{1}{x^2}.$$

In class we have seen that $\lim_{x\to\infty} \frac{1}{x^2} = 0$, and therefore also $\lim_{x\to\infty} -\frac{1}{x^2} = -0 = 0$. Since these limits are the same, by the squeeze theorem $\lim_{x\to\infty} \frac{\sin(x)}{x^2}$ is equal to this common limit, i.e., $\lim_{x\to\infty} \frac{\sin(x)}{x^2} = 0$.