1. The table is:

x	0	1	2	3	4	5	6	7	8	9
$f^{-1}(x)$	9	8	7	6	5	4	3	2	1	0
$(f^{-1})'(x)$	-11	$-\frac{1}{3}$	$-\frac{9}{2}$	$-\frac{3}{5}$	$-\frac{1}{8}$	$-\frac{1}{5}$	$-\pi$	$-\frac{3}{2}$	-2	-1

In order to fill out the table, we just need to use these two rules:

(i) To figure out $f^{-1}(x)$ we ask "what number should be put into f to get out x?" The answer to the question is the value of $f^{-1}(x)$.

(ii) To figure out $(f^{-1})'(x)$ we use the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))},$$

valid as long as $f'(f^{-1}(x)) \neq 0$, which doesn't happen in this table.

2.

(a) Suppose that $x_\circ$ is a value of x where $a(x_\circ) = 0$. Plugging this into (i) we would get $(0)^2 - b(x_\circ)^2 = 1$, or $b(x_\circ)^2 = -1$. This is impossible since $b(x_\circ)^2$ must be greater than or equal to zero (like the square of any real number). Therefore there is no value of $x_\circ$ for which $a(x_\circ) = 0$.

(b) Rewriting (i) we get $a(x)^2 = b(x)^2 + 1$. If $g(x)$ is any function then, since $a(x)$ and $b(x)$ have domain all of $\mathbb{R}$, it makes sense to compose a and b with g. Plugging in $g(x)$ for x we obtain $a(g(x))^2 = b(g(x))^2 + 1$.

(c) By (iii) $b(x)$ is injective. By the theorem on existence and differentiablity of inverse functions from class, b has an inverse function.

(d) The key point of the definition of inverse function is that $b(b^{-1}(x)) = x$. Using $b^{-1}(x)$ in place of $g(x)$ in part (b), we get $a(b^{-1}(x))^2 = b(b^{-1}(x))^2 + 1 = x^2 + 1$.

(e) By the theorem on existence and differentiability of inverse functions from class, if the original function is differentiable, then f^{-1} is differentiable at an inverse function $f^{-1}(x)$ all points x for which $f'(f^{-1}(x)) \neq 0$. Let's apply this to the function $b(x)$ to see that b^{-1} is differentiable on all of $\mathbb{R}$.

We know that b is differentiable. By (*v*) $b'(x) = a(x)$. By part (a) $a(x)$ is never zero. Therefore, there is no $x \in \mathbb{R}$ for which $a(b^{-1}(x)) = 0$, and so $b^{-1}(x)$ is differentiable on all of $\mathbb{R}$.

(*f*) If an inverse function f^{-1} is differentiable at a point x, its derivative is given by the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

(That is also part of the theorem on the existence and differentiability of inverse functions.)

(REMINDER: the inverse function is differentiable at x when $f'(f^{-1}(x)) \neq 0$.)

Let's apply this to the function $b(x)$. By (*v*) $b'(x) = a(x)$, and so the formula above reads

$$(b^{-1})'(x) = \frac{1}{a(b^{-1}(x))}.$$

By part (d) we know that $a(b^{-1}(x))^2 = x^2 + 1$, and substituting $b^{-1}(x)$ in place of x in (*ii*) we know that $a(b^{-1}(x)) \leqslant 0$. Putting these two facts together we get $a(b^{-1}(x)) = -\sqrt{x^2+1}$.

Combining this with the formula above we obtain $(b^{-1})'(x) = \dfrac{-1}{\sqrt{x^2+1}}$, a formula which depends only on x.

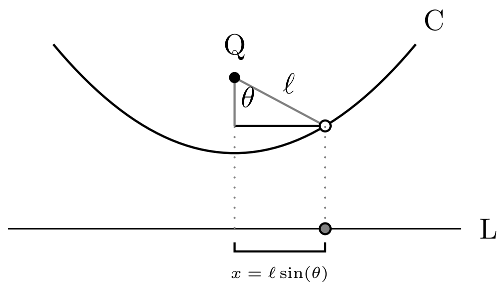
NOTE: The functions $a(x)$ and $b(x)$ really exist. They are

$$\begin{aligned} a(x) &= -\frac{e^{-x}+e^{x}}{2}, \text{ and} \\ b(x) &= \frac{e^{-x}-e^{x}}{2}. \end{aligned}$$

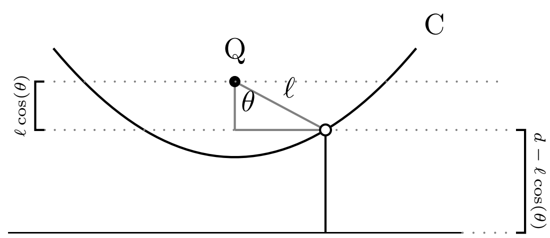
It is easy to check all the properties, except for perhaps (*iii*) from the description above. To see (*iii*), that $b(x)$ is injective, note that the derivative $b'(x) = a(x)$ is never zero (by using the argument from part (a) above). This means that $b'(x) < 0$ for all x, and so $b(x)$ is a (strictly) decreasing function. Then if x_1 and x_2 are any numbers with $x_1 < x_2$, we have $b(x_1) > b(x_2)$. In particular $b(x_1) \neq b(x_2)$, and so b is injective.

3.

(*a*) By drawing the triangle we see that $x(t) = \ell(t)\sin(\theta(t))$.



(*b*) The distance from the object to the line L is the distance from Q to the line, minus the distance that the object is below Q, or $d - \ell(t)\cos(t)$. The distance from point Q to the object is $\ell(t)$ (this is the definition of $\ell(t)$!).



The equation expressing the condition that these are equal is therefore

$$\ell(t) = d - \ell(t)\cos(t)$$

i.e., $\ell(t) + \ell(t)\cos(\theta(t)) = d$, or $d = \ell(t)\left(1 + \cos(\theta(t))\right)$.

(*c*) Starting with the relationships in (a) and (b) and differentiating with respect to the variable t, we get

$$\begin{aligned} \frac{dx}{dt} &= \sin(\theta(t))\frac{d\ell}{dt} + \ell(t)\cos(\theta(t))\frac{d\theta}{dt} \\ 0 &= \left(1+\cos(\theta(t))\right)\frac{d\ell}{dt} - \ell(t)\sin(\theta(t))\frac{d\theta}{dt} \end{aligned}$$

At time $t_\circ$ this is

$$
\begin{aligned}
5 &= \frac{\sqrt{3}}{2}\frac{d\ell}{dt} + 12 \cdot \frac{1}{2}\frac{d\theta}{dt} \\
0 &= \frac{3}{2}\frac{d\ell}{dt} - 12 \cdot \frac{\sqrt{3}}{2}\frac{d\theta}{dt}
\end{aligned}
$$

which we can write in matrix form as

$$
\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 6 \\ \frac{3}{2} & -6\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{d\ell}{dt} \\ \frac{d\theta}{dt} \end{bmatrix}.
$$

The inverse of the 2×2 matrix above is

$$
\begin{bmatrix} \frac{\sqrt{3}}{2} & 6 \\ \frac{3}{2} & -6\sqrt{3} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{3} \\ \frac{1}{12} & -\frac{1}{12\sqrt{3}} \end{bmatrix}.
$$

Multiplying on the left by this inverse we find that

$$
\begin{bmatrix} \frac{d\ell}{dt} \\ \frac{d\theta}{dt} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{3} \\ \frac{1}{12} & -\frac{1}{12\sqrt{3}} \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{3}} \\ \frac{5}{12} \end{bmatrix},
$$

and so at time $t_\circ$, $\dfrac{d\ell}{dt} = \dfrac{5}{\sqrt{3}}$ m/s and $\dfrac{d\theta}{dt} = \dfrac{5}{12}$ rad/s.