1.

(*a*) Let $f(x) = \left(3+\sin(x)\right)^{\left(e^{x^2}\right)}$. Then $\ln(f(x)) = e^{x^2}\cdot\ln\left(3+\sin(x)\right)$, and so

$$\frac{f'(x)}{f(x)} = \tfrac{d}{dx}\ln(f(x)) = \tfrac{d}{dx}e^{x^2}\cdot\ln\left(3+\sin(x)\right) = 2xe^{x^2}\ln\left(3+\sin(x)\right)+e^{x^2}\cdot\tfrac{1}{3+\sin(x)}\cdot(0+\cos(x)).$$

Multiplying through by $f(x)$ gives

$$f'(x) = \left(3+\sin(x)\right)^{\left(e^{x^2}\right)}\cdot e^{x^2}\cdot\left(2x\ln\left(3+\sin(x)\right)+\tfrac{\cos(x)}{3+\sin(x)}\right).$$

(*b*) Let $g(x) = \left(2^x+3x^4\right)^{(\ln x)}$. Then $\ln(g(x)) = \ln(x)\cdot\ln\left(2^x+3x^4\right)$, and so

$$\frac{g'(x)}{g(x)} = \tfrac{d}{dx}\ln(g(x)) = \tfrac{d}{dx}\ln(x)\cdot\ln\left(2^x+3x^4\right) = \frac{1}{x}\cdot\ln\left(2^x+3x^4\right)+\ln(x)\cdot\frac{\ln(2)\cdot 2^x+12x^3}{2^x+3x^4}.$$

Multiplying by $g(x)$ gives

$$g'(x) = \left(2^x+3x^4\right)^{(\ln x)}\cdot\left(\frac{1}{x}\cdot\ln\left(2^x+3x^4\right)+\ln(x)\cdot\frac{\ln(2)\cdot 2^x+12x^3}{2^x+3x^4}.\right)$$

2. If $f(x) = \dfrac{x-1}{x^2+15}$ then

$$f'(x) = \frac{(1)(x^2+15)-(x-1)(2x)}{(x^2+15)^2} = \frac{-x^2+2x+15}{(x^2+1)^2} = -\frac{x^2-2x-15}{x^2+15}.$$

We first find the critical points, i.e., those points where $f'(x)=0$. This is the same as solving $x^2-2x-15=(x+3)(x-5)=0$, with solutions $x=-3$ and $x=5$, both of which are in the interval $[-9,9]$

Now we just need to check the value of $f(x)$ at the critical points and at the endpoints $x=-9$ and $x=9$.

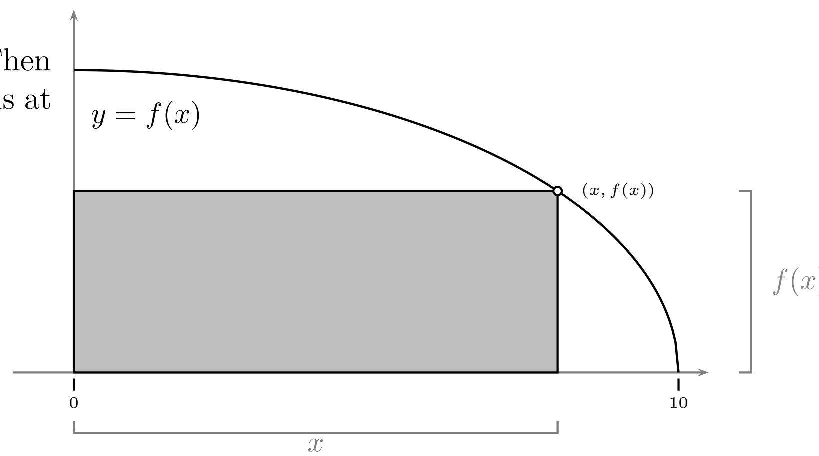
x	-9	-3	5	9
$f(x)$	$-\frac{5}{48}$	$-\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{12}$

From the table, the maximum value occurs at $x = 5$ where $f(5) = \frac{1}{10}$, and the minimum value occurs at $x = -3$ where $f(-3) = -\frac{1}{6}$.

3. Let x be the width of the rectangle. Then the corner of the rectangle on the graph is at the point $(x, f(x))$.

The area of the rectangle is therefore

$$A(x) = \frac{1}{2} \cdot x \cdot \sqrt{100 - x^2}.$$



Differentiating, we have

$$A'(x) = \tfrac{1}{2}\left(\sqrt{100 - x^2} - \frac{2x \cdot x}{2\sqrt{100 - x^2}}\right) = \frac{(100 - x^2) - x^2}{2\sqrt{100 - x^2}} = \frac{100 - 2x^2}{2\sqrt{100 - x^2}}.$$

So $A'(x) = 0$ when $100 - x^2 = x^2$, or $x^2 = 50$. Thus $x = \sqrt{50}$ is the only critical point in $[0, 10]$. The endpoints are $x = 0$ and $x = 10$. Testing all the points, we have

$$A(0) = 0, \qquad A(10) = 0, \qquad \text{and } A(\sqrt{50}) = \tfrac{1}{2}\sqrt{50} \cdot \sqrt{50} = 25.$$

Therefore the maximum area of such a rectangle is 25.

4. Starting with $f(x) = 2\arctan(\sqrt{x}) - \arcsin\left(\frac{x-1}{x+1}\right)$, we have

(*a*) $f(0) = 2\arctan(\sqrt{0}) - \arcsin\left(\frac{0-1}{0+1}\right) = 2\arctan(0) - \arcsin(-1) = 2 \cdot 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$.

(*b*) $$f'(x) = \frac{2}{1 + (\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{1 - \left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{(1)(x+1) - (x-1)(1)}{(x+1)^2}$$

$$= \frac{1}{\sqrt{x}(x+1)} - \frac{1}{\sqrt{\frac{(x+1)^2 - (x-1)^2}{(x+1)^2}}} \cdot \frac{2}{(x+1)^2} = \frac{1}{\sqrt{x}(x+1)} - \frac{1}{\sqrt{\frac{4x}{(x+1)^2}}} \cdot \frac{2}{(x+1)^2}$$

$$= \frac{1}{\sqrt{x}(x+1)} - \frac{(x+1)}{2\sqrt{x}} \cdot \frac{2}{(x+1)^2} = \frac{1}{\sqrt{x}(x+1)} - \frac{1}{\sqrt{x}(x+1)} = 0.$$

(*c*) Since $f'(x) = 0$ we know that f is a constant function on $\{x \in \mathbb{R} \mid x \geqslant 0\}$ (we saw in class that this is a consequence of the mean value theorem). Since $f(0) = \frac{\pi}{2}$ this constant is $\frac{\pi}{2}$, i.e. $f(x) = 2\arctan(\sqrt{x}) - \arcsin\left(\frac{x-1}{x+1}\right) = \frac{\pi}{2}$ for all $x \geqslant 0$. Rearranging, this means that $2\arctan(\sqrt{x}) = \arcsin\left(\frac{x-1}{x+1}\right) + \frac{\pi}{2}$ for all $x \geqslant 0$.

5.

(*a*) We have $f(0) = 0+\sin(0) = 0$, and $f(4\pi) = 4\pi+\sin(4\pi) = 4\pi+0 = 4\pi$. Therefore the average rate of change of f on the interval $[0, 4\pi]$ is

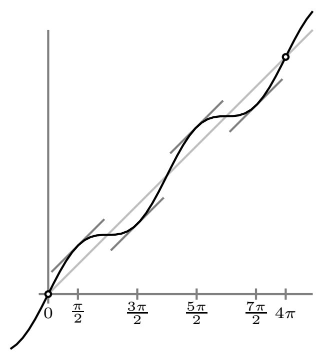
$$\frac{f(4\pi) - f(0)}{4\pi - 0} = \frac{4\pi - 0}{4\pi - 0} = 1.$$

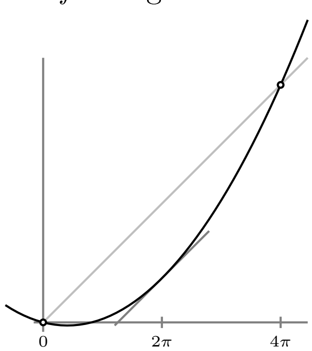
(*b*) For the function g we also have $g(0) = 0 + \frac{1}{10} \cdot 0 \cdot (0 - 4\pi) = 0$ and $g(4\pi) = 4\pi + \frac{1}{10} \cdot 4\pi \cdot (4\pi - 4\pi) = 4\pi$, and so the average rate of change of g on the interval $[0, 4\pi]$ is also

$$\frac{g(4\pi) - g(0)}{4\pi - 0} = \frac{4\pi - 0}{4\pi - 0} = 1.$$

(*c*) Since $f'(x) = 1 + \cos(x)$, the solutions c to the equation $1 = f'(c) = 1 + \cos(c)$ are those c such that $\cos(c) = 0$. There are four solutions in the interval $(0, 4\pi)$, namely, $c = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, and $\frac{7\pi}{2}$.

The points and tangent lines are marked on the graph of f at right.





(*d*) Similarly, since $g'(x) = 1 + \frac{1}{10} \cdot (2x - 4\pi) = 1 + \frac{x-2\pi}{5}$, the solutions c to the equation $1 = g'(c) = 1+\frac{x-2\pi}{5}$ are those c so that $c-2\pi = 0$.

Clearly the unique solution is $c = 2\pi$. The point and corresponding tangent line are marked on the graph of g at left.

(*e*) No.

(*f*) Let us look at the argument step by step, and see which parts make sense, and which do not.

> *Since $f(a) = g(a)$ and $f(b) = g(b)$, the average rates of change of f and g on $[a, b]$ are the same.*

This part of the argument is correct. We can easily write out the calculation which shows us why this claim is true. Since $f(a) = g(a)$ and $f(b) = g(b)$, we certainly do have

$$\frac{f(b) - f(a)}{b - a} = \frac{g(b) - g(a)}{b - a}.$$

> *Let m be this average rate of change.*

There is nothing wrong with this either. In a mathematical argument we are allowed to give names to different numbers or functions we are interested in to make it easier to refer to them later.

> *By the mean value theorem there is a point $c \in (a, b)$ so that $f'(c) = m$.*

This is also correct – the MVT does guarantee at least one c in (a, b) with $f'(c)$ equal to the average rate of change of f on $[a, b]$.

> *Also by the mean value theorem there is a point $c \in (a, b)$ so that $g'(c) = m$.*

This is true too. This is the same argument with f, but applied to g.

> *Therefore $f'(c) = m = g'(c)$, and so $f'(c) = g'(c)$.*

Here is where things go wrong. The problem is that the c guaranteed by the MVT so that $f'(c) = m$ does not have to be the same c guaranteed by the MVT so that $g'(c) = m$. We used the same name each time, but that does not mean that they are the same number (rather, it means that we are being careless!).

We can see this with the example in (a)–(e). For f, the possible c's were $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, and $\frac{7\pi}{2}$, while for g the only possible c was 2π. None of the c's which work for f also work for g.

Instead, it would have been clearer to give different names to these two values, and write :

> *By the mean value theorem there is a point $c_1 \in (a, b)$ so that $f'(c_1) = m$. Also by the mean value theorem there is a point $c_2 \in (a, b)$ so that $g'(c_2) = m$.*

Now the mistake is clear. If we say

> *Therefore $f'(c_1) = m = g'(c_2)$, and so $f'(c_1) = g'(c_2)$.*

that is a completely correct statement. But, all we have shown is that there are c_1 and c_2 so that $f'(c_1) = g'(c_2)$. We have not shown the existence of a single number c so that $f'(c) = g'(c)$.

(*g*) Since f and g are continuous, so is $h(x) = f(x) - g(x)$. Similarly, since f and g are differentiable on (a, b) so is h. Since $f(a) = g(a)$ we have $h(a) = f(a) - g(a) = f(a) - f(a) = 0$. Similarly, since $f(b) = g(b)$ we have $h(b) = 0$.

Therefore the average rate of change of h on the interval $[a, b]$ is

$$\frac{h(b) - h(a)}{b - a} = \frac{0 - 0}{b - a} = 0.$$

By the MVT there is a $c \in (a, b)$ so that $h'(c) = 0$ (i.e., so that $h'(c)$ is equal to the average rate of change of h on $[a, b]$). For this c, we have

$$0 = h'(c) = f'(c) - g'(c),$$

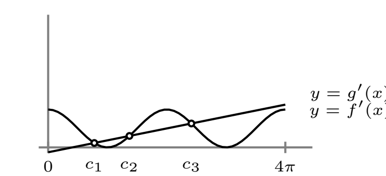
and therefore $f'(c) = g'(c)$. □

Extended Note. It is interesting to go back to our example from (*a*)–(*e*) and look for points c so that $f'(c) = g'(c)$. To do this, we need to solve the equation

$$1 + \cos(c) = 1 + \frac{1}{10} \cdot (2c - 4\pi) = 1 + \frac{c - 2\pi}{5},$$

or $\cos(c) = \frac{c-2\pi}{5}$.

This equation does not have a simple solution (e.g, one which we can get by standard algebraic manipulations). At right are the graphs of the functions $f'(x) = 1 + \cos(x)$ and $g'(x) = 1 + \frac{x-2\pi}{5}$ on the interval $[0, 4\pi]$. From the intersections of the graphs we can see that there are three points where $f'(c) = g'(c)$.



Numerical methods give these three solutions as approximately

$$\begin{aligned} c_1 &\approx 2.4457182006805377571\ldots \\ c_2 &\approx 4.3058022778507454938\ldots \\ c_3 &\approx 7.5896253155490974367\ldots. \end{aligned}$$

At right are the graphs of f and g, along with the tangent lines to each graph at each of the three points above.

You can see that for each c_i, $f'(c_i) = g'(c_i)$, although when $i \neq j$, the slopes over c_i are different from the slopes over c_j.