1. L'Hôpital's rule gives us a way to try and calculate limits of indeterminate forms.

(a) $\lim_{x\to\infty} \frac{\ln(x)}{2\sqrt{x}}$. This is an indeterminate form of the type $\boxed{\frac{\infty}{\infty}}$, and so L'Hôpital's rule applies. Therefore

$$\lim_{x\to\infty} \frac{\ln(x)}{2\sqrt{x}} \overset{\text{L'Hôp}}{=\!=\!=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{\sqrt{x}}{x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0.$$

(b) $\lim_{x\to 1^+} \left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right)$. This is an indeterminate form of the type $\boxed{\infty - \infty}$. In order to apply L'Hôpital's rule, we bring everything over a common denominator.

$$\lim_{x\to 1^+} \left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) = \lim_{x\to 1^+} \frac{(x-1)-\ln(x)}{(x-1)\ln(x)}.$$

This limit is an indeterminate form of the type $\boxed{\frac{0}{0}}$ and so L'Hôpital's rule applies. Therefore

$$\begin{aligned}\lim_{x\to 1^+} \left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) &= \lim_{x\to 1^+} \frac{(x-1)-\ln(x)}{(x-1)\ln(x)} \overset{\text{L'Hôp}}{=\!=\!=} \lim_{x\to 1^+} \frac{1-0-\frac{1}{x}}{\ln(x)+(x-1)\frac{1}{x}} \\ &= \lim_{x\to 1^+} \frac{x-1}{x\ln(x)+(x-1)}\end{aligned}$$

This limit is again an indeterminate form of the type $\boxed{\frac{0}{0}}$, so we can apply L'Hôpital's rule again:

$$\lim_{x\to 1^+} \frac{x-1}{x\ln(x)+(x-1)} \overset{\text{L'Hôp}}{=\!=\!=} \lim_{x\to 1^+} \frac{1-0}{\ln(x)+x\cdot\frac{1}{x}+(1-0)} = \lim_{x\to 1^+} \frac{1}{\ln(x)+2} = \frac{1}{2}.$$

Therefore $\lim_{x\to 1^+} \left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) = \frac{1}{2}$.

(c) $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x$ where $c\in\mathbb{R}$ is a fixed constant. This is an indeterminate form of the type $\boxed{1^\infty}$. In order to apply L'Hôpital's rule we need to do a little rearranging first.

Let $L=\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x$. Then

$$\ln L=\ln\left(\lim_{x\to\infty}\left(1+\tfrac{c}{x}\right)^x\right)=\lim_{x\to\infty}\ln\left(1+\tfrac{c}{x}\right)^x=\lim_{x\to\infty}x\ln\left(1+\tfrac{c}{x}\right).$$

The middle equality (bringing the limit from the inside of ln to the outside) holds because ln is a continuous function (this was the theorem on limits and continuous functions, from the class on Oct 1st).

This last limit is of the form $\boxed{\infty\cdot 0}$, which is still an indeterminate form, but not quite one where we can use L'Hôpital's rule. Writing x as $\frac{1}{\frac{1}{x}}$ we can rewrite the last limit as

$$\lim_{x\to\infty}x\ln\left(1+\tfrac{c}{x}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}}$$

which has the form $\boxed{\frac{0}{0}}$ and so we can finally apply L'Hôpital's rule. Applying the rule, we have

$$\ln L=\lim_{x\to\infty}\frac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}}\overset{\text{L'Hôp}}{=\!=}\lim_{x\to\infty}\frac{\frac{1}{1+\frac{c}{x}}\cdot\frac{-c}{x^2}}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{c}{1+\frac{c}{x}}=\frac{c}{1+0}=c.$$

Therefore $L=e^{\ln L}=e^c$, i.e., $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x=e^c$.

(d) $\lim_{x\to 0}\dfrac{e^x-1-x-\frac{x^2}{2}}{x^3}$. This is an indeterminate form of the type $\boxed{\frac{0}{0}}$. Applying L'Hôpital's rule (three times) we have

$$\lim_{x\to 0}\frac{e^x-1-x-\frac{x^2}{2}}{x^3}\overset{\text{L'Hôp}}{=\!=}\lim_{x\to 0}\frac{e^x-1-x}{3x^2}\overset{\text{L'Hôp}}{=\!=}\lim_{x\to 0}\frac{e^x-1}{6x}\overset{\text{L'Hôp}}{=\!=}\lim_{x\to 0}\frac{e^x}{6}=\frac{1}{6}.$$

We could keep applying L'Hôpital's rule because each of the intermediate limits was also of the form $\boxed{\frac{0}{0}}$. Therefore $\lim_{x\to 0}\dfrac{e^x-1-x-\frac{x^2}{2}}{x^3}=\dfrac{1}{6}$.

(e) $\lim_{x\to 0} \frac{\arctan(x) - x}{x^3}$. This limit is again of the indeterminate form $\boxed{\frac{0}{0}}$. Applying L'Hôpital's rule once we get:

$$\lim_{x\to 0} \frac{\arctan(x) - x}{x^3} \overset{\text{L'Hôp}}{=\!=\!=} \lim_{x\to 0} \frac{\frac{1}{1+x^2} - 1}{3x^2} = \lim_{x\to 0} \frac{\frac{1}{1+x^2} - \frac{1+x^2}{1+x^2}}{3x^2} = \lim_{x\to 0} \frac{-x^2}{3x^2(1+x^2)} = -\frac{1}{3}.$$

2. A function is called *increasing* on an interval $[a, b]$ if whenever $x_1,\ x_2 \in [a, b]$ and $x_1 < x_2$ then $f(x_1) \leqslant f(x_2)$. We've proved in class that if f is differentiable on (a, b) and $f'(x) \geqslant 0$ for all $x \in (a, b)$ then f is increasing on $[a, b]$.

We similarly know that if $f'(x) > 0$ for all $x \in (a, b)$ then f is *strictly increasing*, i.e. if $x_1,\ x_2 \in (a, b)$ and $x_1 < x_2$ then $f(x_1) < f(x_2)$.

To prove the inequalities, or strict inequalities, below, it is therefore sufficient to check that one of these two conditions holds (or in one of the cases, the corresponding statement for decreasing functions).

(a) Suppose that $f(x) = \sin(x) + x$. Then f is defined on all of $\mathbb{R}$. Since $f'(x) = \cos(x) + 1 \geqslant 0$ for all $x \in \mathbb{R}$, this means that f is increasing on all of $\mathbb{R}$. Therefore if $x_1 < x_2$, then $\sin(x_1) + x_1 = f(x_1) \leqslant f(x_2) = \sin(x_2) + x_2$.

(b) Suppose that $f(x) = 3x^2 + \sin^2(x^2) + 1$; the domain of f is all of $\mathbb{R}$. The derivative of f is

$$\begin{aligned} f'(x) &= 6x + 2\sin(x^2)\cos(x^2) \cdot (2x) = 6x + 4x\sin(x^2)\cos(x^2) \\ &= x(6 + 4\sin(x^2)\cos(x^2)). \end{aligned}$$

Since both sin and cos are always between -1 and $+1$, the product $\sin(x^2)\cos(x^2)$ is also between -1 and $+1$ (we can even be more precise: Since $\sin(x^2)\cos(x^2) = \frac{1}{2}\sin(2x^2)$, the product $\sin(x^2)\cos(x^2)$ is always between $-\frac{1}{2}$ and $\frac{1}{2}$). Therefore $6 + 4\sin(x^2)\cos(x^2) > 0$ for all x. If we restrict to $x > 0$, then $x(6+4\sin(x^2)\cos(x^2)) > 0$ for all $x > 0$, i.e., $f'(x) > 0$ for all $x > 0$.

This shows that f is strictly increasing on $(0, \infty)$, so if $0 < x_1 < x_2$ then $f(x_1) < f(x_2)$, i.e., $3x_1^2 + \sin^2(x_1^2) + 1 < 3x_2^2 + \sin^2(x_2^2) + 1$. Since both numbers are positive, when we take reciprocals we get

$$\frac{1}{3x_1^2 + \sin^2(x_1^2) + 1} > \frac{1}{3x_2^2 + \sin^2(x_2^2) + 1}.$$

Alternate Solution

Suppose that $f(x) = \frac{1}{3x^2+\sin^2(x)+1}$. Then

$$f'(x) = \frac{-x(6+4\sin(x^2)\cos(x^2))}{(3x^2+\sin^2(x^2)+1)^2}$$

By the argument in the first solution, the numerator in f' is always negative when $x > 0$ (the numerator is the derivative from the first solution times -1). Therefore f is decreasing on the interval $(0,\infty)$, so if $0 < x_1 < x_2$ then $f(x_1) \geqslant f(x_2)$, i.e.,

$$\frac{1}{3x_1^2+\sin^2(x_1^2)+1} > \frac{1}{3x_2^2+\sin^2(x_2^2)+1}.$$

(*c*) Let $f(x) = x\ln x$. Then $f'(x) = \ln(x) + x\cdot\frac{1}{x} = \ln(x)+1$. Since $\ln(\frac{1}{e}) = -1$, if $x > \frac{1}{e}$ then $\ln(x) > -1$ (here we are using the fact that $\ln(x)$ is a strictly increasing function on $(0,\infty)$), so when $x > \frac{1}{e}$ then $f'(x) > 0$, i.e., f is strictly increasing on $(\frac{1}{e},\infty)$. Therefore if $\frac{1}{e} < x_1 < x_2$ we have that $x_1\ln(x_1) < x_2\ln(x_2)$.

(*d*) Let $f(x) = e^x$. Since $f'(x) = e^x > 0$ for all x, e^x is an increasing function on $\mathbb{R}$. Therefore if $y_1 < y_2$, then $e^{y_1} < e^{y_2}$.

(*e*) If $\frac{1}{e} < x_1 < x_2$ then by part (c) $x_1\ln(x_1) < x_2\ln(x_2)$. By part (d) (with $y_1 = x_1\ln(x_1)$ and $y_2 = x_2\ln(x_2)$) this means that $e^{x_1\ln(x_1)} < e^{x_2\ln(x_2)}$.

Alternate solution:

Let $f(x) = e^{x\ln(x)}$. Then $f'(x) = e^{x\ln(x)}\cdot(\ln(x)+x\cdot\frac{1}{x}) = e^{x\ln(x)}(\ln(x)+1)$. Since $e^y > 0$ for any y (and in particular for $y = x\ln(x)$), and since, by the argument in part (c), $\ln(x)+1 > 0$ when $x > \frac{1}{e}$, f is increasing on $(\frac{1}{e},\infty)$. Therefore, if $\frac{1}{e} < x_1 < x_2$, $e^{x_1\ln(x_1)} < e^{x_2\ln(x_2)}$.

(*f*) Since $x\ln(x) = \ln(x^x)$, $e^{x\ln(x)} = e^{\ln(x^x)} = x^x$. By part (e) we have $e^{x_1\ln(x_1)} < e^{x_2\ln(x_2)}$, and therefore $x_1^{x_1} < x_2^{x_2}$.

Alternate solution:

Let $f(x) = x^x$. We would like to calculate $f'(x)$ to see if f is increasing. It's not so clear how to differentiate f. However, if we look at $\ln f(x) = \ln(x^x) = x\ln(x)$, and differentiate both sides, we get

$$\frac{f'(x)}{f(x)} = \ln(x)+1,$$

so $f'(x) = f(x)(\ln(x)+1) = x^x(\ln(x)+1)$. When $x > 0$, $x^x > 0$, and when $x > \frac{1}{e}$ $\ln(x)+1 > 0$, so when $x > \frac{1}{e}$, $f'(x) > 0$. Therefore $f(x)$ is increasing on $(\frac{1}{e},\infty)$, and so if $\frac{1}{e} < x_1 < x_2$ then $x_1^{x_1} < x_2^{x_2}$.

3.

(a) If $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$, and $f''(x) = \frac{2}{x^3}$. Therefore $f''(x)$ is > 0 when $x > 0$, and so f is convex (i.e., concave up) on $(0, \infty)$.

(b) Since f is convex on $(0, \infty)$, Jensen's inequalites hold in the following form : for any x_1, $x_2 \in (0, \infty)$, and any r_1, $r_2 \geqslant 0$ with $r_1 + r_2 = 1$, we have

$$f(r_1 x_1 + r_2 x_2) \leqslant r_1 f(x_1) + r_2 f(x_2).$$

(More generally, such an inequality holds for $x_1, \ldots, x_k \in (0, \infty)$, and $r_1, \ldots, r_k \geqslant 0$ with $\sum_{i=1}^{k} r_i = 1$, but for this problem we only need the case $k = 2$.)

In particular, choosing $r_1 = \frac{1}{3}$, $r_2 = \frac{2}{3}$, we have

$$\frac{1}{\frac{1}{3}x_1 + \frac{2}{3}x_2} = f\left(\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2\right) \leqslant \frac{1}{3}f(x_1) + \frac{2}{3}f(x_2) = \frac{1}{3x_1} + \frac{2}{3x_2},$$

which is the inequality we want to show.

(c) Let $f(x) = x\ln(x)$. Then $f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$, and $f''(x) = \frac{1}{x}$. For $x > 0$, we have $f''(x) > 0$, so f is a convex function. As in (b), this means that $f(r_1 x_1 + r_2 x_2) \leqslant r_1 f(x_1) + r_2 f(x_2)$ for all x_1, $x_2 \in (0, \infty)$, and r_1, $r_2 \geqslant 0$ with $r_1 + r_2 = 1$. Picking $r_1 = \frac{2}{5}$ and $r_2 = \frac{3}{5}$, this means that we have

$$\left(\frac{2x_1 + 3x_2}{5}\right)\ln\left(\frac{2x_1 + 3x_2}{5}\right) = f\left(\tfrac{2}{5}x_1 + \tfrac{3}{5}x_2\right) \leqslant \tfrac{2}{5}f(x_1) + \tfrac{3}{5}f(x_2) = \tfrac{2}{5}x_1\ln(x_1) + \tfrac{3}{5}x_2\ln(x_2).$$

(d) Let $f(x) = \ln(x)$. Then $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$, so that $f''(x) < 0$ for $x \in (0, \infty)$. Therefore f is concave (equivalently, concave down). This means that for any $x_1, \ldots, x_k \in (0, \infty)$, and any $r_1, \ldots, r_k \geqslant 0$ with $\sum_{i=1}^{k} r_i = 1$ we have the inequality

$$f\left(r_1 x_1 + r_2 x_2 + \cdots + r_k x_k\right) \geqslant r_1 f(x_1) + r_2 f(x_2) + \cdots + r_k f(x_k).$$

If we take $k = n$, and $r_1 = r_2 = \cdots = r_n = \frac{1}{n}$, this is

$$f\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) \geqslant \frac{1}{n}\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right),$$

or

$$\ln\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) \geqslant \frac{1}{n}\left(\ln(x_1) + \ln(x_2) + \cdots + \ln(x_n)\right).$$

When $x_1 = 1$, $x_2 = 2$, ..., $x_n = n$ (i.e., choosing $x_i = i$ for $i = 1, \ldots, n$) this is the inequality

$$\ln\left(\frac{1}{n}(1+2+\cdots+n)\right) \geqslant \frac{1}{n}\left(\ln(1)+\ln(2)+\cdots+\ln(n)\right).$$

Using the formula that $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$, the left hand side of the inequality above is $\ln(\frac{n+1}{2})$. Multiplying both sides of the inequality by n then gives

$$n\ln\left(\frac{n+1}{2}\right) \geqslant \ln(1)+\ln(2)+\cdots+\ln(n),$$

which is the inequality we wanted to show.

Finally, since e^x is an increasing function, if we apply e^x to both sides, it preserves the inequality, and we get

$$n! = 1\cdot 2\cdot 3\cdots(n-1)\cdot n = e^{\ln(1)+\ln(2)+\cdots+\ln(n)} \leqslant e^{n\ln(\frac{n+1}{2})} = \left(\frac{n+1}{2}\right)^n.$$

Extended Remark: In analysis (the branch of mathematics to which Calculus belongs), one is often interested in the asymptotic growth of a function, as some parameter gets very large, or very small. The estimate above, $n! \leqslant \left(\frac{n+1}{2}\right)^n$ is correct, but can be greatly improved.

A better approximation is provided by *Stirling's formula* :

$$n! \sim \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n \qquad\qquad (\text{as } n\to\infty).$$

For instance, when $n = 100$, the upper bound $\left(\frac{n+1}{2}\right)^n$ is about $2.133721886 \times 10^{170}$, while $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ is about $9.324847782 \times 10^{157}$. In other words, when $n = 100$ Stirling's approximation gives an estimate which is 10^{13} (i.e., about 10 trillion) times better than our upper bound. For larger n, the improvement provided by Stirling's formula is even better.

(*e*) Let $f(x) = \sqrt{x^2+1}$. Then $f'(x) = \frac{1}{2\sqrt{x^2+1}}\cdot(2x) = \frac{x}{\sqrt{x^2+1}}$, and

$$f''(x) = \frac{1\cdot\sqrt{x^2+1} - x\cdot\frac{x}{\sqrt{x^2+1}}}{(\sqrt{x^2+1})^2} = \frac{(x^2+1)-x^2}{(\sqrt{x^2+1})^3} = \frac{1}{(x^2+1)^{\frac{3}{2}}}.$$

Thus, $f''(x)$ is > 0 when $x > 0$, so f is convex on $(0,\infty)$. As in part (*d*) for x_1, $x_2, \ldots, x_n \in (0,\infty)$, and $r_1 = r_2 = \cdots = r_n = \frac{1}{n}$, this gives us the estimate

$$f\left(\frac{1}{n}(x_1+x_2+\cdots+x_n)\right) \leqslant \frac{1}{n}\left(f(x_1)+f(x_2)+\cdots+f(x_n)\right).$$

(The direction of the inequality is the opposite from (d), since in (d) the function was concave, and here it is convex.) As in (d), taking $x_i = i$ for $i = 1, \ldots, n$ (i.e., taking $x_1 = 1$, $x_2 = 2$, …, $x_n = n$), and using the formula $1+2+\cdots+n = \frac{1}{2}n(n+1)$, this is

$$f\left(\tfrac{1}{2}(n+1)\right) \leqslant \frac{1}{n}\left(f(1) + f(2) + \cdots + f(n)\right), \quad \text{or}$$

$$\sqrt{(\tfrac{1}{2}(n+1))^2 + 1} \leqslant \frac{1}{n}\left(\sqrt{1^2+1} + \sqrt{2^2+1} + \cdots + \sqrt{n^2+1}\right).$$

Multiplying through by n, and simplifying $(\frac{n+1}{2})^2 + 1$ this gives

$$\sqrt{1^2+1} + \sqrt{2^2+1} + \cdots + \sqrt{n^2+1} \geqslant n \cdot \sqrt{\frac{n^2+2n+5}{4}} = \frac{n}{2}\sqrt{n^2+2n+5}.$$

Note: Unlike the previous inequality, involving $n!$, I know of no application of the inequality above. It was included as an example of one of the many things one can prove using convexity.

4.

(a) To show that $f(g(x))$ is concave up using inequalities, we need to show that

$$f(g(r_1x_1 + r_2x_2)) \leqslant r_1 f(g(x_2)) + r_2 f(g(x_2))$$

for any x_1, x_2 in $\mathbb{R}$, and r_1, $r_2 \in \mathbb{R}$ with r_1, $r_2 \geqslant 0$ and $r_1 + r_2 = 1$.

Let's first write down what we know about f and g. For any r_1, r_2 and x_1, x_2 as above :

($a1$) $g(r_1x_1 + r_2x_2) \leqslant r_1 g(x_1) + r_2 g(x_2)$, since g is concave up.

($a2$) Since f is increasing, for any y_1, y_2, with $y_1 \leqslant y_2$ we have $f(y_1) \leqslant f(y_2)$.

($a3$) $f(r_1z_1 + r_2z_2) \leqslant r_1 f(z_1) + r_2 f(z_2)$ for any z_1, $z_2 \in \mathbb{R}$, (and any r_1, r_2 as above) since f is concave up.

Then, starting with inequality ($a1$):

$$g(r_1x_1 + r_2x_2) \leqslant r_1 g(x_1) + r_2 g(x_2)$$

if we apply f to both sides (which preserves the inequality by ($a2$)) we get

$$f(g(r_1x_1 + r_2x_2)) \leqslant f(r_1 g(x_1) + r_2 g(x_2))$$

and now applying inequality ($a3$) with $z_1 = g(x_1)$, and $z_2 = g(x_2)$, to the right hand side we get

$$f(r_1 g(x_1) + r_2 g(x_2)) \leqslant r_1 f(g(x_1)) + r_2 f(g(x_2)).$$

Combining the last two inequalities gives us

$$f(g(r_1 x_1 + r_2 x_2)) \leqslant r_1 f(g(x_1)) + r_2 f(g(x_2))$$

which shows that $f(g(x))$ is concave up.

(b) To show that $f(g(x))$ is concave up using the second derivative, we need to show that the second derivative of $f(g(x))$ is greater than or equal to zero for all x.

Let's first write down what we know about f and g.

($b1$) $g''(x) \geqslant 0$ for all x since g is concave up.

($b2$) $f'(x) \geqslant 0$ for all x since f is increasing.

($b3$) $f''(x) \geqslant 0$ for all x since f is concave up.

The second derivative of $f(g(x))$ is (using the chain rule and the product rule when differentiating twice) :

$$f''(g(x))\,(g'(x))^2 + f'(g(x))g''(x).$$

Now $f''(g(x)) \geqslant 0$ by ($b3$), $(g'(x))^2$ is $\geqslant 0$ since the value for any x is the square of a real number, $f'(g(x))$ is $\geqslant 0$ by ($b2$), and $g''(x)$ is $\geqslant 0$ by ($b1$). Therefore all the terms in the expression above are $\geqslant 0$, and so $f''(g(x))\,(g'(x))^2 + f'(g(x))g''(x) \geqslant 0$ too, and this shows that $f(g(x))$ is concave up.

Alternate Solution

The other characterization of concave up by derivatives is that the first derivative is *increasing*. The first derivative of $f(g(x))$ is $f'(g(x))g'(x)$ (by the chain rule), so to show that $f(g(x))$ is concave up, we need to show that if $x_1 < x_2$ then $f'(g(x_1))g'(x_1) \leqslant f'(g(x_2))g'(x_2)$. (We could also show that $f'(g(x))g'(x)$ is increasing by showing that its first derivative is positive; this is the same as the solution above).

Let's write down what we know about f and g:

($b4$) $g'(x)$ is an increasing function since g is concave up.

($b5$) $f'(x) \geqslant 0$ for all x since f is increasing.

($b6$) $f'(x)$ is an increasing function since f is concave up.

We have to be slightly careful in demonstrating the inequality. The function $g(x)$ might be decreasing on some part of the real line, so $x_1 < x_2$ might not imply that $g(x_1) \leqslant g(x_2)$. The function g switches from being decreasing to being increasing when $g'(x_0) = 0$. This can happen for at most one point $x_0 \in \mathbb{R}$ since $g'(x)$ is an increasing function. Suppose that there is a point x_0 where $g'(x_0) = 0$. We consider two cases:

Case I: Suppose that both x_1 and x_2 are greater than x_0. Then g is increasing on $[x_0, \infty)$, so if $x_1 < x_2$ then $g(x_1) \leqslant g(x_2)$. Since f' is increasing, this also gives us $f'(g(x_1)) \leqslant f'(g(x_2))$, or $\frac{f'(g(x_2))}{f'(g(x_1))} \geqslant 1$. Since g' is increasing, we have $g'(x_1) \leqslant g'(x_2)$, and we know that both $g'(x_1)$ and $g'(x_2)$ are positive, since x_1, and x_2 are greater than x_0. Since $g'(x_2)$ is positive, multiplying it by a number greater than one makes it larger, and so

$$g'(x_1) \leqslant g'(x_2) \leqslant g'(x_2) \cdot \frac{f'(g(x_2))}{f'(g(x_1))}.$$

Multiplying through by the positive number $f'(g(x_1))$ gives

$$f'(g(x_1))g'(x_1) \leqslant f'(g(x_2))g'(x_2),$$

which is what we wanted to show.

Case II: Suppose that both x_1 and x_2 are less than x_0. Then g is decreasing on $(-\infty, x_0]$ so if $x_1 < x_2$ then $g'(x_1) \geqslant g'(x_2)$. Since f' is increasing, this gives us $f'(g(x_1)) \geqslant f'(g(x_2))$ or $\frac{f'(g(x_1))}{f'(g(x_2))} \geqslant 1.$. Since g' is increasing, we have $g'(x_1) \leqslant g'(x_2)$, and we know that both $g'(x_1)$ and $g'(x_2)$ are negative since x_1 and x_2 are less than x_0. Since $g'(x_1)$ is negative, multiplying it by a number greater than one makes it smaller, and so

$$g'(x_1) \cdot \frac{f'(g(x_1))}{f'(g(x_2))} \leqslant g'(x_1) \leqslant g'(x_2).$$

Multiplying through by the positive number $f'(g(x_2))$ gives

$$f'(g(x_1))g'(x_1) \leqslant f'(g(x_2))g'(x_2),$$

which is what we wanted to show.

Finally, since case I shows that $f'(g(x))g'(x)$ is increasing on $[x_0, \infty)$, and case II shows that $f'(g(x))g'(x)$ is increasing on $(-\infty, x_0]$, the function $f'(g(x))g'(x)$ must be increasing on $(-\infty, \infty)$, so the derivative of $f(g(x))$ is increasing, and therefore $f(g(x))$ is concave up.

Note that if there is no number x_0 such that $g'(x_0) = 0$, then either $g'(x) > 0$ for all x in $\mathbb{R}$ or $g'(x) < 0$ for all $x \in \mathbb{R}$, and so either case I (if $g'(x) > 0$) or case II (if $g'(x) < 0$) is sufficient by itself to establish that $f'(g(x))g'(x)$ is an increasing function on all of $\mathbb{R}$.