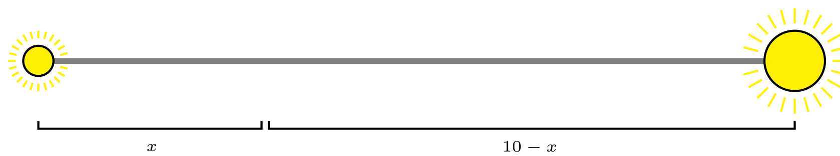
1. Let x be the distance from the weaker light source, and $k > 0$ the constant of proportionality (the value of k does not change the answer to the problem, but it is important that k is positive).



The illumination from the weaker source is then $I_1 = \frac{k}{x^2}$, while the illumination from the stronger source is $I_2 = \frac{4k}{(10-x)^2}$. The total illumination is therefore

$$I(x) = I_1 + I_2 = \frac{k}{x^2} + \frac{4k}{(10-x)^2},$$

and we want to find the maximum on the interval $(0, 10)$ (the function is not defined at the endpoints).

We start by looking for the critical points :

$$\frac{dI}{dx} = \frac{-2k}{x^3} + \frac{-8k}{(10-x)^3}(-1) = 2k\left(\frac{4}{(10-x)^3} - \frac{1}{x^3}\right).$$

The derivative is zero when $\frac{4}{(10-x)^3} = \frac{1}{x^3}$, or (taking cube roots) when $\frac{\sqrt[3]{4}}{10-x} = \frac{1}{x}$. Cross multiplying we get the equation $10 - x = \sqrt[3]{4}x$, with solution $x = \frac{10}{1+\sqrt[3]{4}}$.

Now let us answer the question : Is this critical point a local max, a local min, an absolute max, an absolute min, or neither?

The second derivative of I is

$$\frac{d^2I}{dx^2} = 2k\left((-3)\cdot\frac{4}{(10-x)^4}\cdot(-1) - (-3)\cdot\frac{1}{x^4}\right) = 6k\left(\frac{4}{(10-x)^4} + \frac{1}{x^4}\right).$$

From the formula, we can see that $I''(x)$ is > 0 on $(0, 10)$, and thus I' is increasing on $(0, 10)$. Let $x_0 = \frac{10}{1+\sqrt[3]{4}}$ be the critical point. Since $I'(x_0) = 0$ the fact that I' is increasing means that $I'(x) \geqslant 0$ for $x \in (x_0, 10)$, and $I'(x) \leqslant 0$ for $x \in (0, x_0)$. Thus, I is decreasing on $(0, x_0)$ and increasing on $(x_0, 10)$, and we conclude that x_0 is an absolute min of I on $(0, 10)$.

Therefore the total illumination is the weakest at the point $\frac{10}{1+\sqrt[3]{4}}$m from the weaker light source, or equivalently $\frac{10\sqrt[3]{4}}{1+\sqrt[3]{4}}$m from the stronger light source.

Alternate arguments for x_0 being the absolute min

Here are some other possible arguments that x_0 is the absolute min.

(1) As $x \to 0^+$ or $x \to 10^-$, $I(x) \to \infty$. Therefore, I has to have an absolute min somewhere in $(0, 10)$. This absolute min will also be a local min, and therefore appear on any list of critical points. Since there is only one critical point, this critical point must be the absolute min.

(2) Instead of using I'' to understand the sign of I', we could argue directly : We know that

$$0 = I'(x_0) = 2k\left(\frac{4}{(10-x_0)^3} - \frac{1}{x_0^3}\right).$$

For $x > x_0$, taking the reciprocal and cubing we get $-\frac{1}{x^3} > -\frac{1}{x_0^3}$ (since, for instance, cubing, being an increasing function, preserves the direction of the inequality, taking the reciprocal reverses it, and multiplying by -1 reverses the inequality again). If $x > x_0$ then we also have $10 - x < 10 - x_0$, and going through the same procedure gives $\frac{1}{(10-x)^3} > \frac{1}{(10-x_0)^3}$. Thus, for $x > x_0$, each term of $I'(x)$ is strictly greater than the corresponding term in $I'(x_0)$, and therefore $I'(x) > I'(x_0) = 0$, and so I is increasing on $(x_0, 10)$.

Similarly, starting with $x < x_0$, we get $-\frac{1}{x^3} < \frac{1}{x_0}^3$ and $\frac{1}{(10-x)^3} < \frac{1}{(10-x_0)^3}$, and therefore $I'(x) < I'(x_0) = 0$ for $x \in (0, x_0)$. This shows that I is decreasing in $(0, x_0)$, and putting both statements together (I decreasing on $(0, x_0)$; I increasing on $(x_0, 10)$) we conclude as above that x_0 is an absolute min of I on $(0, 10)$.

2. By the Pythagorean theorem, the distance between two points (x_1, y_2) and (x_2, y_2) in $\mathbb{R}^2$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

(a) The distance from the junction to site A is $\sqrt{(x-0)^2 + (0-4)^2} = \sqrt{x^2 + 16}$. The distance from the junction to site B is the same.

The distance from the junction to site C is $|10 - x|$. We can see this in two ways : The entire semester we have been using the fact that the distance from x_1 to x_2 on the x-axis is $|x_2 - x_1|$. Or, we can use the Pythagorean theorem again, and calculate the distance as $\sqrt{(10-x)^2 + (0-0)^2} = \sqrt{(10-x)^2} = |10 - x|$.

But, it is clear that the minimum must occur when $x \in [0, 10]$. Putting the junction at a point $(x, 0)$ with $x > 10$ clearly results in a longer total distance than just putting the junction at $(10, 0)$. Similarly, putting the junction at $(x, 0)$ with $x < 0$ clearly results in a longer total distance than putting the junction at $(0, 0)$. So, the minimum must occur with $x \in [0, 10]$. Since we know that $x \leqslant 10$, we know that $|10 - x| = 10 - x$.

Adding up the distances, the total distance we want to minimize is

$$T(x) = 2\sqrt{x^2+16} + 10 - x.$$

Taking the derivative we have

$$T'(x) = 2 \cdot \frac{2x}{2\sqrt{x^2+16}} - 1 = \frac{2x}{\sqrt{x^2+16}} - 1.$$

Therefore, $T'(x) = 0$ when $\frac{2x}{\sqrt{x^2+16}} = 1$, or $2x = \sqrt{x^2+16}$. If we square both sides, we get the equation $4x^2 = x^2 + 16$, with solutions $x = \pm\sqrt{\frac{16}{3}} = \pm\frac{4}{\sqrt{3}}$.

The equation we are trying to solve is $2x = \sqrt{x^2+16}$, and the right hand side of this equation is always $\geqslant 0$. Therefore, only the solution $x = \frac{4}{\sqrt{3}}$ is correct. The extra solution of $x = -\frac{4}{\sqrt{3}}$ was introduced when we squared both sides of the equation.

To find the minimum of $T(x)$ on $[0, 10]$ it is enought to check the endpoints and the critical point.

x	0	$\frac{4}{\sqrt{3}}$	10
$T(x)$	18	$4\sqrt{3}+10$	$4\sqrt{29}$

$\approx$

x	0	$\frac{4}{\sqrt{3}}$	10
$T(x)$	18	$16.92820\ldots$	$21.540659\ldots$

So the minimum occurs when $x = \frac{4}{\sqrt{3}}$.

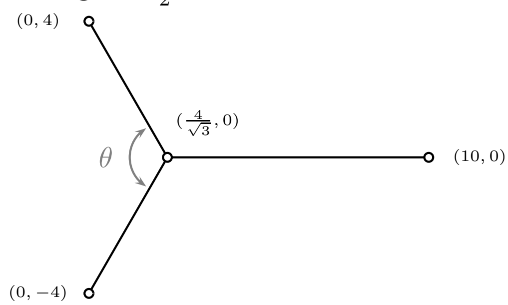
Another way to see that putting the junction at $(\frac{4}{\sqrt{3}}, 0)$ gives the minimum total distance is to consider the sign of $T'(x)$. For that, it is easier to write $T'(x)$ like this :

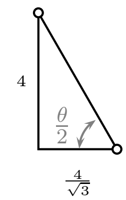
$$T'(x) = \frac{2x}{\sqrt{x^2+16}} - 1 = \frac{2x - \sqrt{x^2+16}}{\sqrt{x^2+16}} = \frac{2x - \sqrt{x^2+16}}{\sqrt{x^2+16}} \cdot \frac{2x + \sqrt{x^2+16}}{2x + \sqrt{x^2+16}}$$

$$= \frac{(2x)^2 - (\sqrt{x^2+16})^2}{(\sqrt{x^2+16})(2x + \sqrt{x^2+16})} = \frac{3x^2 - 16}{(\sqrt{x^2+16})(2x + \sqrt{x^2+16})}.$$

The denominator in the last expression is always positive, while the numerator is negative for $x \in [0, \frac{4}{\sqrt{3}})$ and positive for $x \in (\frac{4}{\sqrt{3}}, 10]$. Therefore $T(x)$ is decreasing on the first interval, and increasing on the second interval, so that the global minimum occurs at $x = \frac{4}{\sqrt{3}}$.

(b) Below left is a picture of the junction at the spot which minimizes the total distance, along with one of the angles, θ. Below right is a picture of the right triangle where one angle is $\frac{\theta}{2}$.





From the picture, $\tan(\frac{\theta}{2}) = \frac{4}{\frac{4}{\sqrt{3}}} = \sqrt{3}$, so

$$\frac{\theta}{2} = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3},$$

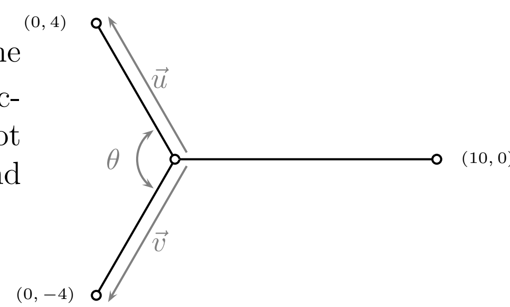
and therefore $\theta = \frac{2\pi}{3}$ (aka $120°$).

By symmetry the other two angles in the picture are equal, and add up to $2\pi - \frac{2\pi}{3} = 2 \cdot \frac{2\pi}{3}$. Therefore each of those angles is also $\frac{2\pi}{3}$ radians.

That is, at this minimizing point, all angles are equal.

Alternate Solution: Here is a second solution, using the dot-product from linear algebra.

Let $\vec{u} = (-\frac{4}{\sqrt{3}}, 4)$ be the vector connecting the junction to site A, and $\vec{v} = (-\frac{4}{\sqrt{3}}, -4)$ be the vector connecting the junction to site B. By the dot product formula, the cosine of angle between $\vec{u}$ and $\vec{v}$ is



$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|u\| \cdot \|v\|} = \frac{\frac{16}{3} - 16}{\left(\sqrt{(-\frac{4}{\sqrt{3}})^2 + (4)^2}\right)\left(\sqrt{(-\frac{4}{\sqrt{3}})^2 + (-4)^2}\right)} = \frac{\frac{16}{3} - 16}{\left(\frac{16}{3} + 16\right)}$$

$$= \frac{\frac{1}{3} - 1}{\left(\frac{1}{3} + 1\right)} = \frac{-\frac{2}{3}}{\frac{4}{3}} = -\frac{1}{2}.$$

Therefore $\theta = \arccos(-\frac{1}{2}) = \frac{2\pi}{3}$ radians, as before.

(c) With the point $(10, 0)$ moved to $(2, 0)$, the new total distance function is

$$T(x) = 2\sqrt{x^2 + 16} + 2 - x.$$

The derivative of this function is the same as last time :

$$T'(x) = 2 \cdot \frac{2x}{2\sqrt{x^2 + 16}} - 1 = \frac{2x}{\sqrt{x^2 + 16}} - 1.$$

Therefore the critical point is also the same, $x = \frac{4}{\sqrt{3}}$.

But, since $\frac{4}{\sqrt{3}} \approx 2.309401077\ldots > 2$, the critical point is outside the interval $[0, 2]$. Now, as in part (a) we can proceed in two different ways to determine the minimum of T on $[0, 2]$. For variety, let us do them in the opposite order from question (a).

By examining the sign of the derivative. Since the derivative is the same as last time, we again know that $T'(x)$ is negative on $[0, \frac{4}{\sqrt{3}})$. But $[0, 2] \subset [0, \frac{4}{\sqrt{3}})$, so $T'(x)$ is negative on all of $[0, 2]$. This means that T is a decreasing function on $[0, 2]$, and hence the minimum value is found at $x = 2$.

By checking the value of T at the endpoints.

x	0	2
$T(x)$	10	$4\sqrt{5}$

$\approx$

x	0	2
$T(x)$	10	$8.944271908\ldots$

Then it is also clear that the minimum value occurs when $x = 2$, with minimum value $T(2) = 4\sqrt{5}$.

3. Let $f(x) = xe^{-x^2}$.

A: (*Domain*) The domain of f is all of $\mathbb{R}$.

B: (*Intercepts*) If $x = 0$, $f(0) = 0$ so the y-intercept is 0. Solving $0 = f(x) = xe^{-x^2}$, since e^{-x^2} is never zero we can divide to get $x = 0$, i.e., the only x-intercept is $x = 0$, so the graph of f crosses the axes only at $(0, 0)$.

C: (*Asymptotes*) Since $f(x)$ is a continuous function, defined on all of $\mathbb{R}$, it has no vertical asymptotes (for any a, $\lim_{x \to a} f(x) = f(a)$ since f is continuous and defined for all a therefore the limit couldn't be $\pm\infty$, and so f can't have any vertical asymptotes).

To evaluate $\lim_{x\to\infty} xe^{-x^2}$, it is helpful to know the relative rates of growths of some standard functions as $x \to \infty$:

$$\ln(x) \text{ grows slower than } x, \text{ which grows slower than } x^2, \text{ which grows slower than } x^3, \ldots,$$

$$\ldots, \text{ which grows slower than } x^n, \ldots, \text{ which grows slower than } e^x, \ldots$$

Here “grows slower than” means that (for example) the limit of $\frac{\ln(x)}{x}$ as $x \to \infty$ is 0 ($\ln(x)$ and x both go to ∞ as $x \to \infty$, but $\ln(x)$ grows so much slower than x does, that the limit is 0).

The function e^x grows faster than any power of x, so that for any $n > 0$, $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$.

Since e^{x^2} grows even faster than e^x as $x \to \infty$, we conclude that $\lim x \to \infty \frac{x}{e^{x^2}} = 0$.

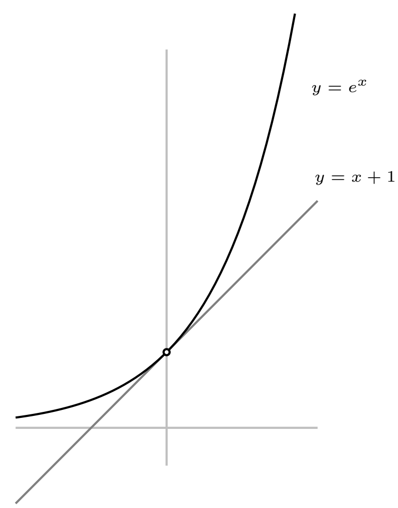
Alternate arguments for the Horizontal Asymptote.

(1) We can also use L'Hôpital's rule to evaluate $\lim_{x\to\pm\infty} xe^{-x^2}$ (the limit is an indeterminate form of the type $\boxed{0 \cdot \infty}$). If we rewrite xe^{-x^2} as $\frac{x}{e^{x^2}}$ we get an indeterminate form of the type $\boxed{\frac{\infty}{\infty}}$ and so L'Hôpital's rule applies.

Therefore we have $\lim\limits_{x\to\infty} \dfrac{x}{e^{x^2}} \overset{\text{L'Hôp}}{=\!=} \lim\limits_{x\to\infty} \dfrac{1}{2xe^{x^2}} = 0$ and $\lim\limits_{x\to-\infty} \dfrac{x}{e^{x^2}} \overset{\text{L'Hôp}}{=\!=} \lim\limits_{x\to-\infty} \dfrac{1}{2xe^{x^2}} = 0$, so $x = 0$ is the only horizontal asymptote.

(2) The function e^x is convex (its second derivative is e^x again, which is always positive). An alternate method of argument uses a property of convexity not mentioned in class, but much like the defining secant property.

For a convex function, the tangent line to any point of the graph always lies below the graph. (For a concave function, the tangent line is always above the graph.) So, for instance the tangent line at the point $(0, 1)$ is always below the graph of e^x (as shown in the picture at right) :



Since the equation of the tangent line is $y = x + 1$ (the tangent line has slope 1, and passes through $(0, 1)$, this gives us the inequality :

$$1 + x \leqslant e^x$$

valid for all $x \in \mathbb{R}$. In particular, for $x \geqslant 0$ we get the inequalities

$$0 \leqslant x \leqslant 1 + x \leqslant e^x.$$

Dividing by e^{x^2} gives

$$0 \leqslant \frac{x}{e^{x^2}} \leqslant \frac{e^x}{e^{x^2}} = e^{x-x^2}.$$

As $x \to \infty$, e^{x-x^2} goes to 0, and hence by the squeeze theorem we conclude that

$$\lim_{x\to\infty} \frac{x}{e^{x^2}} = 0.$$

D: (*Symmetry*) The function f is an *odd* function: $f(-x) = (-x)e^{-(-x)^2} = -(xe^{-x^2}) = -f(x)$.

E: (*Increasing/Decreasing*) The first derivative is

$$f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = (1-2x^2)e^{-x^2}.$$

Since e^{-x^2} is always positive, the sign of $f'(x)$ is the same as the sign of $(1-2x^2)$. Therefore $f'(x) > 0$ when $x^2 < \frac{1}{2}$ or $|x| < \frac{1}{\sqrt{2}}$, and $f'(x) < 0$ when $x^2 > \frac{1}{2}$ or $|x| > \frac{1}{\sqrt{2}}$. I.e., f is increasing on the interval $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$, and decreasing on the intervals $(-\infty, -\frac{1}{\sqrt{2}}]$ and $[\frac{1}{\sqrt{2}}, \infty)$.

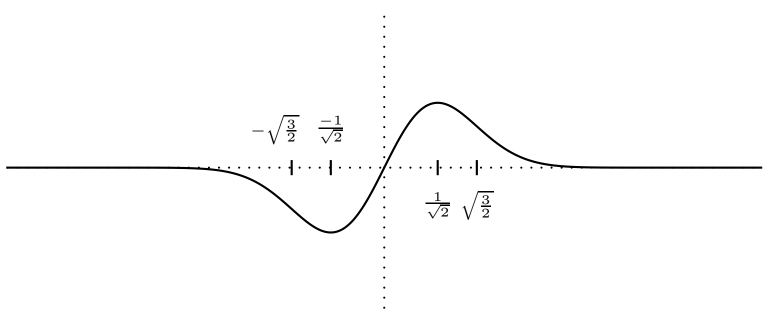
F: (*Critical points*) The critical points occur at $x = \pm\frac{1}{\sqrt{2}}$. When $x = \frac{1}{\sqrt{2}}$ the function f is changing from increasing to decreasing. Therefore $x = \frac{1}{\sqrt{2}}$ is a local max. When $x = -\frac{1}{2}$ the function f is changing from decreasing to increasing. Therefore $x = -\frac{1}{\sqrt{2}}$ is a local min.

G: (*Concavity/Inflection points*) The second derivative is

$$f''(x) = (-4x)e^{-x^2} + (1-2x^2)e^{-x^2}(-2x) = (-6x+4x^3)e^{-x^2} = 2x(2x^2-3)e^{-x^2}.$$

The second derivative is zero when $x = 0$ or $x = \pm\sqrt{\frac{3}{2}}$. The second derivative is positive (and the graph of f concave up) on the intervals $[-\sqrt{\frac{3}{2}}, 0]$ and $[\sqrt{\frac{3}{2}}, \infty)$. The second derivative is negative (and the graph of f concave down) on the intervals $(-\infty, -\sqrt{\frac{3}{2}}]$ and $[0, \sqrt{\frac{3}{2}}]$. The graph changes concavity at $x = 0$, $x = \pm\sqrt{\frac{3}{2}}$.

H: (*Graph*) Here is a sketch of the graph of f:



4. Let $f(x) = axe^{-bx}$ where a, $b > 0$.

(a) Since $\frac{df}{dx} = ae^{-bx} + ax(-b)e^{-bx} = a(1 - bx)e^{-bx}$, there is a unique critical point at $x = 1/b$. The first derivative is positive and the function is increasing on $(-\infty, 1/b)$; the first derivative is negative and the function is decreasing on $(1/b, \infty)$. Therefore, the point $\left(\frac{1}{b}, \frac{a}{be}\right)$ is a global maximum.

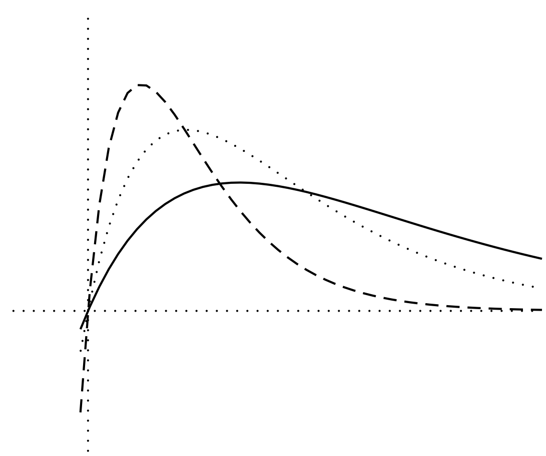
We have $\frac{d^2f}{dx^2} = a(-b)e^{-bx} + a(1 - bx)(-b)e^{-bx} = -ab(2 - bx)e^{-bx}$. Thus, the second derivative is positive and the function is concave up on $(-\infty, 2/b)$; the second derivative is negative and the s function is concave down on $(2/b, \infty)$. In particular, $x = 2/b$ is the unique inflection point.

(b) Multiplying the function xe^{-bx} by a constant a, stretches the graph vertically if $a > 1$ or shrinks the graph vertically if $0 < a < 1$. In particular, varying the parameter a changes the maximum value of the function.

The parameter b determines the distance from the origin to the global maximum. Specifically, as b increases the maximum value moves closer to the origin.

An even more precise description of how b affects the function is that changing b by a factor of k (i.e., replacing b by kb) contracts the graph in the x-direction and y-directions by a factor of k.

(c) Here are three sample graphs :



(d) By part (a) the maximum occurs at when $x = \frac{1}{b}$ and $y = f(\frac{1}{b}) = \frac{a}{be}$. If we want this to be the point $(2, 3)$, then we need to have $\frac{1}{b} = 2$ and $\frac{a}{be} = 3$, which we solve to get $b = \frac{1}{2}$ and $a = \frac{3}{2}e$.