due date: Sept. 9, 2025

1. Questions about rational and irrational numbers.

(a) If a is rational and b irrational, what can you say about $a+b$? (i.e, is it rational or irrational?) What if a and b are both irrational?

(b) If a is rational and b irrational, what can you say about ab? (*be careful...*).

(c) Is there a number a so that a^2 is irrational, but a^4 is rational?

2. Let $f(x) = |x-3|+1$, $g(x) = \begin{cases} x^2 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$, and $h(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.

Describe and give a rough sketch of the functions

(a) $f(|x|)$ (b) $g(x)+g(-x)$ (c) $g\big(x^2-9\big)$

(d) $h(h(x))$ (e) $h(x)\cdot\sin(x)$

3. Let a, b, c, and d be real numbers, with $a<b$ and $c<d$. In this problem we will prove the statement that

$$\boxed{[a,b] \subseteq [c,d] \text{ if and only if } c \leqslant a \text{ and } b \leqslant d.}$$

The main purpose of the question is to practice writing down a mathematical argument in a clean way. It is also meant to demonstrate a common pattern in mathematical arguments : frequently going back to the definitions of the mathematical objects involved. One reason for this pattern is that the definitions are all we know about the objects.

(a) Write down the definitions of the intervals $[a,b]$ and $[c,d]$ in their"set builder" form. I.e., fill in the following :

$$[a,b] = \Big\{ x \in \mathbb{R} \;\Big|\; \underline{\qquad\qquad} \Big\}$$

and

$$[c,d] = \Big\{ x \in \mathbb{R} \;\Big|\; \underline{\qquad\qquad} \Big\}.$$

As a reminder, here is the the definition of what it means for a set A to be contained in a set B :

$$A \subseteq B \text{ if and only if for all } x \in A,\ x \text{ is also in } B.$$

The claim we want to prove is an “if and only if”. This means that we need to establish two implications. The first is that

$\boxed{c \leqslant a \text{ and } b \leqslant d} \implies \boxed{[a,b] \subseteq [c,d]}$.

By the definition of “one set is included in another”, to show that $[a,b] \subseteq [c,d]$ we need to show that every element of $[a,b]$ is also an element of $[c,d]$. Because of the implication we are trying to show, we are allowed to assume that $c \leqslant a$ and that $b \leqslant d$ while doing so.

So, let x be an element of $[a,b]$. By definition, since $x \in [a,b]$, $a \leqslant x$ and $x \leqslant b$.

Since (by assumption) $c \leqslant a$ and (from above) $a \leqslant x$, we conclude that $c \leqslant x$.

(b) Write out a similar argument to show that, for $x \in [a,b]$, we also have $x \leqslant d$.

Putting these two statements together, we have shown that for any $x \in [a,b]$, we have $c \leqslant x$ and $x \leqslant d$. By the definition of $[c,d]$, this means that $x \in [c,d]$.

Since this argument works for any $x \in [a,b]$, we have passed the test for inclusion of sets, and shown that $[a,b] \subseteq [c,d]$. Now we need to show the implication in the other direction.

$\boxed{[a,b] \subseteq [c,d]} \implies \boxed{c \leqslant a \text{ and } b \leqslant d}$.

This argument is even shorter. Since $a \in [a,b]$, by the assumption that $[a,b] \subseteq [c,d]$ we then know that $a \in [c,d]$. By the definition of $[c,d]$, this means that $c \leqslant a \leqslant d$. In particular $c \leqslant a$.

(c) Write down a similar argument to show that $b \leqslant d$.

Taken together, these arguments establish the reverse implication, and complete the proof.

To make the statement more concrete, let us consider it in two examples.

<u>Example 1</u> ($a = 3$, $b = 5$, $c = 4$, $d = 8$) :

(d) Sketch the intervals $[3,5]$ and $[4,8]$ on the real line.

(e) Is $[3,5] \subseteq [4,8]$?

(f) Which (if any) of the statements $c \leqslant a$ and $b \leqslant d$ are true in this case?

Example 2 ($a = 4$, $b = 7$, $c = 2$, $d = 9$) :

(g) Sketch the intervals $[4, 7]$ and $[2, 9]$ on the real line.

(h) Is $[4, 7] \subseteq [2, 9]$?

(i) Which (if any) of the statements $c \leqslant a$ and $b \leqslant d$ are true in this case?

NOTES ABOUT THE HOMEWORK:

1. You are allowed and encouraged to talk or to work with other people on the homework. However this doesn't mean that you can just copy someone else's work: you are responsible for writing up your own version of the answers.

2. On your homework, before the first question, include a short list of people that you collaborated with while working on the homework. If you didn't talk to anyone, put "collaborators: none". There are no grades deducted for working with other people, but grades *will* be deducted if you don't indicate who you worked with.

3. Write your assignment as neatly as possible. Unlike previous mathematics courses you may have taken, the goal in this course is not just to get the answer, but to understand and to explain the answer clearly. This is especially true when writing up a proof: take the time to sort out your thoughts and organize the argument you are presenting.