Math 120 — Homework Assignment 3

DUE DATE: SEPT. 23, 2025

1. If a, b, A, and B are real numbers with $a \leqslant A$ and $b \leqslant B$, we frequently use the fact that this implies $a + b \leqslant A + B$. Let us briefly justify why. Assuming the inequalities above (i.e., that $a \leqslant A$ and $b \leqslant B$) :

(a) Explain how we know that $a + b \leqslant A + b$.

Here your answer should be quite short — just say which property from the class of Wednesday, September 10th you are using.

(b) Similarly explain how we know that $A + b \leqslant A + B$.

(c) Put (a) and (b) together to show that $a + b \leqslant A + B$.

Now again assume that a, b, A, and B are real numbers with $a \leqslant A$ and $b \leqslant B$. What can we say about ab and AB? As in the case of comparing $\frac{1}{A}$ and $\frac{1}{B}$ from Wednesday's class, the signs of the numbers play a role.

First assume that a, A, b, B are all $\geqslant 0$.

(d) State, with proof, the inequality which holds between ab and AB.

Next, assume that a, A, b, B are all $\leqslant 0$.

(e) State, with proof, the inequality which holds between ab and AB in this case.

Given that $a \leqslant A$, there are three possibilities for the signs of a and A (a and A both positive, both negative, or a negative and A positive). Similarly there are three possibilities for the signs of b and B, for a total of nine possibilities for the signs of a, A, b, and B.

(f) Is there one of these possibilities where it is *not* possible to determine the inequality between ab and AB? (Again assuming that $a \leqslant A$ and $b \leqslant B$ to start with.) If so, state which case, and illustrate this fact with examples.

NOTES: (i) The argument for addition in (a), (b), and (c), may be a useful model for parts of the argument for multiplication. (ii) In (f) "illustrate with examples" means to give an example of a, A, b, and B (with the sign pattern you are illustrating, and satisfying $a \leqslant A$ and $b \leqslant B$) such that $ab < AB$, and give another such example where $ab > AB$.

2. Suppose that $|x-4|<2$. Find upper and lower bounds for

(a) $|x+1|$ (b) $|x-3|$ (c) $|x^2-2x-3|$ (d) $\dfrac{|x-3|}{|x+1|}$.

For parts (a) and (b) give both a geometric argument and an algebraic argument using the triangle inequality. (They really are the same argument and we want to practice seeing that this is true.) You can use both the upper bound and lower bound parts of the triangle inequality in your arguments.

3. Suppose that we know that $|x-6|<\delta$ where δ is some positive number, $\delta<9$.

(a) Find an upper bound (in terms of the unknown number δ) for $\left|\dfrac{5x}{x+4}-3\right|$.

(b) If $|x-6|<4$, what does your answer in (a) give as an upper bound for $\left|\dfrac{5x}{x+4}-3\right|$?

(c) Find a value of δ so that if $|x-6|<\delta$ then $\left|\dfrac{5x}{x+4}-3\right|<\dfrac{1}{2}$.

(d) For any positive integer m, show how to pick δ so that $\left|\dfrac{5x}{x+4}-3\right|<\dfrac{1}{m}$ whenever $|x-6|<\delta$.

(i.e., give a rule for picking δ so that the above inequality is true; the particular δ will depend on m. Part (c) is the case $m=2$).

Note : As part of the problem we are assuming that $\delta<9$, and that should help in some of the steps above.