DUE DATE: SEPT. 30, 2025

1. Find (and give an ϵ–δ proof for) the following limits :

   (a) $\lim_{x\to 6}(x^2+4)$.

   (b) $\lim_{x\to -4}(3x^2+3x+4)$.

   (c) $\lim_{x\to -2}\dfrac{4x-19}{x-7}$.

   (d) $\lim_{x\to 4}\dfrac{3x-2}{2x-3}$.

See the back page of this assignment for a discussion of how to write up the ϵ–δ proofs in question 1.

2. The purpose of this question is to show a more advanced limit (namely $\lim_{x\to c} x^3 = c^3$) directly from the definition (i.e., *not* using any limit theorems).
Let c be any real number.

   (a) Show that $|x^2+cx+c^2| \leqslant |x|^2+|c|\cdot|x|+|c|^2$. (This should be very easy!)

   (b) If $|x-c|<1$, show that $|x|<1+|c|$.

   (c) If $|x-c|<1$, show that $|x^2+cx+c^2|<(1+|c|)^2+|c|(1+|c|)+|c|^2$.

   (d) Use the factorization $x^3-c^3=(x-c)(x^2+cx+c^2)$ to show that if ϵ is some positive number and
   $$|x-c|<\min\left(1, \frac{\epsilon}{(1+|c|)^2+|c|(1+|c|)+|c|^2}\right)$$
   then $|x^3-c^3|<\epsilon$.

   (e) Give an ϵ–δ proof that $\lim_{x\to c} x^3 = c^3$.

3. Use the squeeze theorem to calculate the following limits:

   (a) $\lim_{x\to 0}(x^4+x^2)\sin(1/x)$ (b) $\lim_{x\to 0}(8+x^2\cos(x)\sin(1/x))$

Remarks on ϵ–δ proofs.

An ϵ–δ proof is its own literary form and, like a sonnet or a haiku, has a particular structure. The general shape of such a proof is :

> *Given $\epsilon > 0$, set $\delta =$ (some formula involving ϵ). Then, whenever $0 < |x - c| < \delta$,*
>
> $$|f(x) - L| = \cdots\cdots \text{[some arguments and estimates here]} \cdots\cdots \leqslant \epsilon.$$
>
> *(I.e., some argument which shows that $|f(x) - L| \leqslant \epsilon$ whenever $0 < |x - c| < \delta$.)*
> *Therefore, by the definition of the limit, $\lim_{x \to c} f(x) = L$.* □

The grading scheme for the ϵ–δ proofs on each part of question 1 is as follows :

— Having an initial investigation 2

— Starting the final proof with "Given $\epsilon > 0\ldots$" 1

— Continuing with "set $\delta =$ (some formula)" 1

— Writing "Then if $0 < |x - c| < \delta$" 1

— Using the choice of δ (and possibly an assumption) to bound the other factor 2

— Showing how this implies that $|f(x) - L| \leqslant \epsilon$ 2

— Stating the concluding sentence :
"Therefore, by the definition of limit, $\lim_{x \to c} f(x) = L$. 1

Here, c and L should be the correct values for the problem.
Why insist that the argument is written this way?

(*i*) ϵ–δ proofs do have a strict form, and anyone reading such an argument expects this form to be followed. If you want to show directly that a limit is as claimed, this is the form you need to use.

(*ii*) One of the goals of this class is learning to write a clear mathematical argument (as opposed to writing down a sequence of unexplained and unmotivated calculations). In general it takes extra work to figure out how to organize an argument : Where to start, which notation to introduce, when to introduce it, how to structure the argument.... In contrast, an ϵ–δ proof already comes with its own organizational structure, and so provides an intermediate step between the world of "unexplained calculations" and the world of "clear arguments, organized by the author".

(*iii*) Another goal of the class is to understand the ϵ–δ definition of a limit. The purpose of an ϵ–δ proof is to show that the conditions of that definition are satisfied. So, learning to write such a proof may help in understanding what that definition is saying[1].

[1] This also suggests it might be a good idea to go back and think about that definition again.