DUE DATE: Oct. 21, 2025

1.

(a) Prove that there is some real number x so that $x^{179} + \dfrac{163}{1 + x^2 + \sin^2(x)} = 119$.

(b) Similarly, prove that there is some $x \in \mathbb{R}$ so that $\sin(x) = x - 1$.

Suggestion : Consider using the intermediate value theorem.

2. Let $f(x) = \dfrac{x\,|x| + 3x}{1 + x^2}$, $g(x) = \begin{cases} x \sin(\frac{1}{x^3}) + 2 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$, and $p(x) = f(x)g(x)$.

Find the following derivatives, if they exist.

(a) $f'(0)$ (b) $g'(0)$ (c) $p'(0)$.

Note : You may assume that the absolute value function is a continuous function.

3. Use the formula for the derivative of $\frac{1}{g(x)}$ in terms of the derivative of $g(x)$, and the formula for the derivative of x^n when n is positive, to show that the formula $\frac{d}{dx}x^n = nx^{n-1}$ also holds when n is negative.

For example, here is how that argument would work when $n = -3$:

" To compute the derivative of $f(x) = x^{-3}$, we first write f as $f(x) = \frac{1}{x^3}$. Then, using the quotient rule and the rule for the derivative of x^3, we get :

$$\tfrac{d}{dx}f(x) = -\frac{\frac{d}{dx}x^3}{(x^3)^2} = -\frac{3x^2}{x^6} = -3 \cdot \frac{1}{x^4} = -3x^{-4} = -3x^{-3-1}.$$

So, when $n = -3$, $\frac{d}{dx}x^n = nx^{n-1}$, just like the rule for positive n."

Your job in this question is to write the argument for general negative n.

4. Consider the following functions:

$$f(x) = |9 - x^2| \qquad \text{and} \qquad g(x) = |x - 1|^{1/3}$$

(a) Draw a sketch of the graphs of $f(x)$ and $g(x)$.

(b) Based on your sketch, for which values of x does it seem that $f(x)$ is probably not differentiable?

(c) Similarly, for which values of x does it seem that $g(x)$ is probably not differentiable?

(d) Choose one of the points you listed in (b) and prove that f is not differentiable there by using the definition of differentiability.

(e) Similarly, use the definition of differentiability to see if g is differentiable at the point from (c).

Hint : In class we have seen that a possible reason for which a function is not differentiable at a point is the presence of a corner or a cusp in the graph.