DUE DATE: Nov. 11, 2025

1. Suppose that f is a function differentiable on all of $\mathbb{R}$, that its inverse function f^{-1} exists and is defined on all of $\mathbb{R}$, and that we know the following values for $f(x)$ and $f'(x)$:

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	9	8	7	6	5	4	3	2	1	0
$f'(x)$	-1	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{\pi}$	-5	-8	$-\frac{5}{3}$	$-\frac{2}{9}$	-3	$-\frac{1}{11}$

For the same values of x, fill in the corresponding table of values for $f^{-1}(x)$ and $(f^{-1})'(x)$.

2. Suppose that $a(x)$ and $b(x)$ are two functions whose domain is all of $\mathbb{R}$, and that we know the following six things about them:

(*i*) $\big(a(x)\big)^2 - \big(b(x)\big)^2 = 1$ for all $x \in \mathbb{R}$.

(*ii*) $a(x) \leqslant 0$ for all $x \in \mathbb{R}$.

(*iii*) The function $b(x)$ is an injective (i.e., one to one) function.

(*iv*) $a(x)$ and $b(x)$ are differentiable on all of $\mathbb{R}$.

(*v*) $\frac{d}{dx}\, b(x) = a(x)$.

(*vi*) $\frac{d}{dx}\, a(x) = b(x)$.

Prove or justify each of the statements below. If you use any of the properties of $a(x)$ or $b(x)$ above as part of your argument, indicate which ones you are using. (Or, if you use a theorem from class, state which theorem you are using, and why you know the hypotheses of the theorem hold.)

(*a*) $a(x)$ is never zero. (This means that there is no value $x_\circ$ such that $a(x_\circ) = 0$.)

(*b*) For any function $g(x)$, $a(g(x))^2 = b(g(x))^2 + 1$.
[Here $a(g(x))^2$ means $(a(g(x)))^2$, and similarly for $b(g(x))^2$.]

(*c*) $b(x)$ has an inverse function.

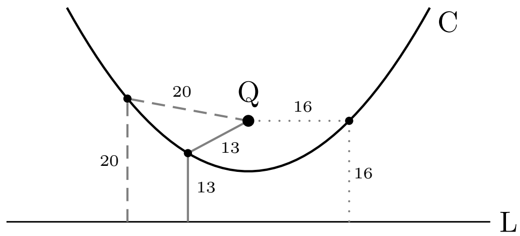
(*d*) Let $b^{-1}(x)$ be the inverse function of $b(x)$. Then $a(b^{-1}(x))^2 = x^2 + 1$.

(*e*) The inverse function $b^{-1}(x)$ is differentiable on all of $\mathbb{R}$.

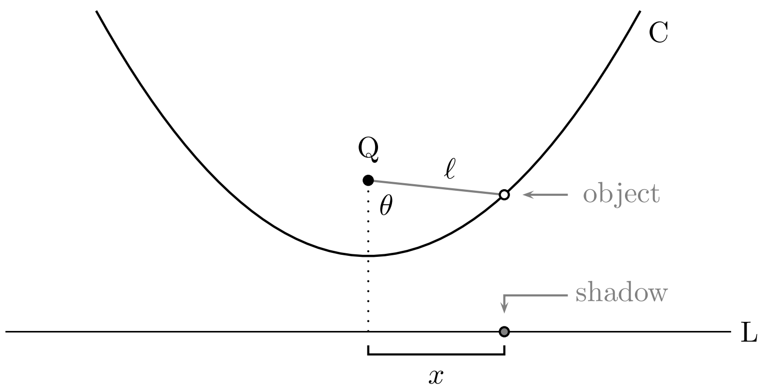
Finally,

(*f*) Find the derivative of $b^{-1}(x)$, and express it as a formula involving only x.

3. A point Q is d metres from a line L. The curve C has the property that the distance from any point on C to Q is the same as the distance from that point to the line L. (The distance will depend on the point.) Here is a picture to show what that means.



An object is travelling along the curve and casts a shadow on L. Let $\ell(t)$ be the distance from Q to the object, $\theta(t)$ the angle that the line connecting Q to the object makes with a vertical line, and $x(t)$ the distance from the shadow to the vertical line, all at time t.



At a certain point $t_\circ$ in time, $\ell(t_\circ) = 12\text{m}$, $\theta(t_\circ) = \frac{\pi}{3}$ radians, and the shadow of the object on L is travelling at 5 m/s to the right.

QUESTION: How fast are ℓ and θ changing at time $t_\circ$?

To answer this,

(*a*) Write down a formula for $x(t)$ involving $\ell(t)$ and $\theta(t)$.

(*b*) Write down an equation involving $\ell(t)$ and $\theta(t)$ which expresses the condition

"*The distance from the object to* Q *is the same as the distance from the object to* L"

(*c*) Use the formulas in (a) and (b) to answer the question above (don't forget the units).