DUE DATE: NOV. 25, 2025

1. Find the following limits:

(*a*) $\lim_{x\to\infty} \frac{\ln(x)}{2\sqrt{x}}$

(*b*) $\lim_{x\to 1^+} \left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right)$

(*c*) $\lim_{x\to\infty} \left(1 + \frac{c}{x}\right)^x$ where $c \in \mathbb{R}$ is a fixed constant.

[SUGGESTION : First take the logarithm of $\left(1+\frac{c}{x}\right)^x$, find the limit of the logarithm, and then exponentiate. Because the logarithm and exponential functions are continuous, we can exchange evaluation of the function and evaluation of the limit.]

(*d*) $\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$

(*e*) $\lim_{x\to 0} \frac{\arctan(x) - x}{x^3}$

2. Prove the following inequalities

(*a*) If $x_1 < x_2$, show that $\sin(x_1) + x_1 \leqslant \sin(x_2) + x_2$.

(*b*) If $0 < x_1 < x_2$, show that $\frac{1}{3x_1^2+\sin^2(x_1^2)+1} > \frac{1}{3x_2^2+\sin^2(x_2^2)+1}$.

(*c*) If $\frac{1}{e} < x_1 < x_2$ show that $x_1 \ln(x_1) < x_2 \ln(x_2)$.

(*d*) if $y_1 < y_2$, show that $e^{y_1} < e^{y_2}$.

(*e*) If $\frac{1}{e} < x_1 < x_2$ show that $e^{x_1 \ln(x_1)} < e^{x_2 \ln(x_2)}$.

(*f*) If $\frac{1}{e} < x_1 < x_2$ show that $x_1^{x_1} < x_2^{x_2}$.

3.

(*a*) Is $f(x) = \frac{1}{x}$ a convex function, a concave function, or neither on the interval $(0, \infty)$?

(b) Prove that for any x_1, $x_2 > 0$ we always have

$$\frac{1}{\frac{1}{3}x_1 + \frac{2}{3}x_2} \leqslant \frac{1}{3x_1} + \frac{2}{3x_2}.$$

Use the method of (a)+(b) to prove the inequalities below.

(c) Prove that for any x_1, $x_2 > 0$, we always have

$$\left(\frac{2x_1 + 3x_2}{5}\right) \ln\left(\frac{2x_1 + 3x_2}{5}\right) \leqslant \tfrac{2}{5}x_1 \ln(x_1) + \tfrac{3}{5}x_2 \ln(x_2).$$

(d) Prove that for any positive integer n we have

$$\ln(1) + \ln(2) + \ln(3) + \cdots + \ln(n) \leqslant n \ln\left(\frac{n+1}{2}\right).$$

Conclude that $n! \leqslant \left(\frac{n+1}{2}\right)^n$.

Suggestion : Consider the function $\ln(x)$, the points $x_1 = 1$, $x_2 = 2$, $\ldots$, $x_n = n$, and $r_1 = r_2 = \cdots = r_n = \frac{1}{n}$. The identity $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ will also be useful.

(e) Prove that for any positive integer n, we have the inequality

$$\sqrt{1^2+1} + \sqrt{2^2+1} + \sqrt{3^2+1} + \sqrt{4^2+1} + \cdots + \sqrt{n^2+1} \geqslant \frac{n}{2}\sqrt{n^2+2n+5}.$$

Suggestion : Consider the function $\sqrt{x^2+1}$, and the x_i and r_i from (d).

4. Suppose that f and g are functions that are defined on all of $\mathbb{R}$, that f is a convex, increasing function, and that g is a convex function. The goal of this problem is to show that the composite function $f(g(x))$ is also convex, in two different ways.

(a) First show this by using the Jensen's inequality definition of convexity, i.e., show that the composite function $f(g(x))$ satisfies the appropriate inequality for any x_1, x_2, and r_1, r_2, with r_1, $r_2 \geqslant 0$ and $r_1 + r_2 = 1$ (equivalently, with λ and $(1-\lambda)$, for $\lambda \in [0,1]$.)

(b) Now suppose that both f and g are twice differentiable. Use the second derivative criterion for convexity to give a second proof that the composite function $f(g(x))$ is convex.

Notes for Question 4: (i) In part (a) your argument shouldn't involve derivatives, since we don't know that f or g are even differentiable. (I.e., the argument for part (a) should work even for non-differentiable convex functions.) (ii) In (b), a good first step is to work out the formula for the second derivative of a composite function $f(g(x))$. (iii) One of the conditions of the problem is that f is increasing. You will need this in parts (a) and (b).