1. The region \( U = \{ (x, y) \mid |y| \leq \sin(x) \} \), consists of infinitely many disconnected "lobes":

Each one associated to an interval \( x \in [2k\pi, (2k+1)\pi] \) on the \( x \)-axis for \( k \in \mathbb{Z} \), i.e., those \( x \) values where \( \sin(x) \geq 0 \). The grey points (those points with \( |y| < \sin(x) \)) are the interior points, and the dark black points (those with \( |y| = \sin(x) \)) are the boundary points.

To solve this problem a good strategy is to use the idea suggested in the second class, namely to first study the case of equality: \( |y| = \sin(x) \). As usual with the absolute value, this breaks up into two cases, \( y = \sin(x) \) if \( y \geq 0 \) and \( -y = \sin(x) \), or \( y = -\sin(x) \) if \( y \leq 0 \).

The set of points where \( y = \sin(x) \) and \( y \geq 0 \) is:

The set of points where \( y = -\sin(x) \) and \( y \leq 0 \) is:

(Note that \( y = \sin(x) \) and \( y \geq 0 \) requires that \( \sin(x) \geq 0 \), while \( y = -\sin(x) \) and \( y \leq 0 \) also requires that \( \sin(x) \geq 0 \), which is where we end up with the interval \([2k\pi, (2k+1)\pi]\). Those are exactly the intervals where \( \sin(x) \geq 0 \).

Putting these together we end up with the set of points where \( |y| = \sin(x) \):
If we now stand on the curve and ask which directions we should move to make the equality into the inequality we want (namely $|y| \leq \sin(x)$), we see that we need to decrease $y$ if we are on the top curve, or increase $y$ on the bottom curve. This leads to the answer given at the beginning.

2. If $u(x, y, t) = e^{-2t} \sin(3x) \cos(2y)$, then

$$u_x(x, y, t) = 3e^{-2t} \cos(3x) \cos(2y),$$
$$u_y(x, y, t) = -2e^{-2t} \sin(3x) \sin(2y), \text{ and}$$
$$u_t(x, y, t) = -2e^{-2t} \sin(3x) \cos(2y).$$

If $u$ is the height of a vibrating membrane (like a drum) above $(x, y)$ at time $t$, then at a point $(x_0, y_0)$ and time $t_0$, $u_t(x_0, y_0, t_0)$ represents how fast the membrane is moving up and down over top the point $(x_0, y_0)$ at time $t_0$. For the other two derivatives, imagine the surface of the membrane at this frozen instant $t_0$, then $u_x(x_0, y_0, t_0)$ is the rate of change of the height of the graph going in the $x$-direction, and $u_y(x_0, y_0, t_0)$ the rate of change going on the $y$-direction.

3. If $F : \mathbb{R}^2 \to \mathbb{R}^3$ is the function $F(x, y) = \left( \sin(\pi x) \cos(\pi y), ye^{xy}, x^2 + y^3 \right)$, then

$$DF(x, y) = \begin{bmatrix} \pi \cos(\pi x) \cos(\pi y) & -\pi \sin(\pi x) \sin(\pi y) \\ ye^{xy} & (1 + xy)e^{xy} \\ 2x & 3y^2 \end{bmatrix}, \text{ so } DF(1, 2) = \begin{bmatrix} -\pi & 0 \\ 4e^2 & 3e^2 \\ 2 & 12 \end{bmatrix}.$$ 

If we head in the direction $\vec{v} = (3, -2)$, this means that the instantaneous rate of change of the three functions is given by

$$DF(1, 1)\vec{v} = \begin{bmatrix} -\pi & 0 \\ 4e^2 & 3e^2 \\ 2 & 12 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3\pi \\ 6e^2 \\ -18 \end{bmatrix}.$$ 

In other words, if we’re at the point $(1, 2)$, and we head off in the direction $(3, -2)$, the instantaneous rate of change of $\sin(\pi x) \cos(\pi y)$ is $-3\pi$, the instantaneous rate of change of $ye^{xy}$ is $6e^2$, and the instantaneous rate of change of $x^2 + y^3$ is $-18$.

4. We’re starting with the function

$$f(x, y) = \begin{cases} \frac{3xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$
(a) If we restrict to the $x$-axis, i.e. points of the form $(x, 0)$, we get the function

$$f(x, 0) = \begin{cases} 
0 & \text{if } (x, 0) \neq (0, 0) \\
x^2 + 0^2 & \text{if } (x, 0) = (0, 0)
\end{cases}$$

which can be more succinctly described by simply saying that $f(x, 0) = 0$ for all values of $x$.

The partial derivative $f_x(0, 0)$ certainly exists. According to the definition of $f_x(0, 0)$ we only need to understand $f(x, 0)$ and see if it has an $x$-derivative at $x = 0$. The zero function (which is what $f(x, 0)$ is) is certainly differentiable — any constant function is. Its derivative, like all constant functions, is 0. Thus $f_x(0, 0) = 0$.

We can also go directly to the definition of the partial derivative:

$$f_x(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0,$$

but it seems simpler just to think about the restriction as above.

(b) Restricting the function $f$ to the $y$-axis, gives $f(0, y) = 0$, just like the restriction to the $x$-axis. For the same reasons, $f_y(0, 0)$ exists and is zero.

(c) If $f$ were differentiable at $(0, 0)$ then its derivative matrix would be

$$Df(0, 0) = \begin{bmatrix} \frac{\partial f}{\partial x}(0, 0) & \frac{\partial f}{\partial y}(0, 0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

(d) For a function $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$, being differentiable at a point $w \in \mathbb{R}^n$ is a promise: a promise that there exists an $m \times n$ matrix $A$ such that, for any vector $\vec{v} \in \mathbb{R}^n$, if we pass through $w$ with velocity vector $\vec{v}$, the instantaneous rates of change of the components of $F$ are given by the vector $A\vec{v}$.

In part (c) we computed what that matrix had to be for the function $f$, by taking the derivative in the $x$- and $y$-directions. Therefore, if $f$ were differentiable at $(0, 0)$ then the instantaneous rate of change in the direction $\vec{v} = (1, 1)$ would be given by

$$Df(0, 0)\vec{v} = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$
(e) If we restrict \( f \) to points of the form \((t, t)\), we get the function

\[
f(t, t) = \begin{cases} 
\frac{3t^3}{t^2 + t^2} & \text{if } (t, t) \neq (0, 0) \\
0 & \text{if } (t, t) = (0, 0)
\end{cases}
\]

or simply the function \( f(t, t) = \frac{3}{2}t \).

As a function of \( t \), this is a line with slope \( \frac{3}{2} \). Its rate of change at \( t = 0 \) is therefore also \( \frac{3}{2} \). You can also write this more formally: \( \frac{\partial}{\partial t} f(t, t) = \frac{3}{2} \), and then plugging in \( t = 0 \) gives \( \frac{3}{2} \).

(f) As explained in part (d), if \( f \) were differentiable, the answers to (d) and (e) would be the same: the rate of change when going through \((0, 0)\) with velocity vector \( \vec{v} = (1, 1) \) would be computed by the matrix \( Df(0, 0) \), giving \( Df(0, 0)\vec{v} = 0 \).

(g) Since the answers to (d) and (e) are different, \( f \) is not differentiable at \((0, 0)\).

It’s worthwhile thinking a bit more about the function \( f \). To try and see what it looks like, let’s restrict it to lines of the form \((v_x t, v_y t)\) for some nonzero vector \( \vec{v} = (v_x, v_y) \):

\[
f(v_x t, v_y t) = \begin{cases} 
\frac{v_x v_y^2 t^3}{(v_x^2 + v_y^2)t^2} & \text{if } (v_x t, v_y t) \neq (0, 0) \\
0 & \text{if } (v_x t, v_y t) = (0, 0)
\end{cases}
\]

or simply \( f(v_x t, v_y t) = \left( \frac{v_x v_y^2}{v_x^2 + v_y^2} \right) t \).

This shows us that the restriction of \( f \) to any direction \( \vec{v} \) is a linear function of \( t \), with \( t \)-slope \( \frac{v_x v_y^2}{v_x^2 + v_y^2} \). Two views of the graph are shown below:

In any direction \( \vec{v} = (v_x, v_y) \), the derivative exists and is equal to \( \frac{v_x v_y^2}{v_x^2 + v_y^2} \). Since this doesn’t depend linearly on \((v_x, v_y)\) this means that \( f \) is not differentiable at \((0, 0)\). This was exactly the kind of example mentioned in class.
5. This time we’re starting with the function $f(x, y) = 25 - x^2 - 2y^2$.

(a) $f(2, 3) = 3$

(b) $g_x(x, y) = m \quad g_x(2, 3) = m$

(c) Clearly we want $m = -4$ and $n = -12$. The graph of $-4x - 12y + c$ passes through $-4(2) - 12(3) + c = -44 + c$ over $(2, 3)$. In order for this to be 3 we need $c = 47$.

(d) $g(2 + tv_x, 3 + tv_y) = -4(2 + tv_x) - 12(3 + tv_y) + 47 = 3 + (-4v_x - 12v_y)t$. This has derivative $-4v_x - 12v_y$ at $t = 0$ (or for any $t$).

(e) $f(2 + tv_x, 3 + tv_y) = 3 + (-4v_x - 12v_y)t - (v_x^2 + 2v_y^2)t^2$. This also has derivative $-4v_x - 12v_y$ when $t = 0$.

(f) The fact that the answers to (d) and (e) are the same is certainly the behaviour we expect from a differentiable function – it’s an indication that $f$ is well approximated by its tangent plane over $(2, 3)$.

This behaviour (that the instantaneous rate of change along lines agrees with that of the plane determined by the partial derivatives) is not enough to guarantee that a function is actually differentiable, although it takes a bit of work to construct an example. (That is, an example where if we go through in straight lines the derivative depends linearly on the velocity vector, but if we go through with other curves it does not. **CHALLENGE:** find one for yourself).

Our function $f$ is differentiable though. For instance, the partial derivatives $f_x(x, y) = -2x$ and $f_y(x, y) = -4y$ are certainly continuous, and we know that’s enough to imply the differentiability of $f$.

(g) Actually, although it may not immediately appear like it, the definition of what it means for $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ to be differentiable is exactly that the separate component functions $F_1, F_2, \text{and} F_3$ be individually differentiable. It is equivalent to the definition of differentiability given in the book.

(h) The derivative of a single function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at a point $(x_1, \ldots, x_n)$ is something which computes the instantaneous rate of change of $f$ in any direction $\vec{v} = (v_1, \ldots, v_n)$, and (as the argument with the tangent planes shows) this association is linear in $\vec{v}$. Therefore, the derivative $DF$ of a function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ at a point $(x_1, \ldots, x_n)$ should be something which computes the $m$ different instantaneous rates of change in a direction $\vec{v}$, and the dependence of the rates of change on $\vec{v}$ should again be linear. But a linear function from $n$-vectors to $m$-vectors is exactly the definition of a linear transformation.