1. Let \( f \) be a function \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \), and \( \mathbf{F} \) and \( \mathbf{G} \) vector fields on \( \mathbb{R}^3 \) (i.e., functions \( \mathbb{R}^3 \rightarrow \mathbb{R}^3 \)). State whether each of the following expressions is a function from \( \mathbb{R}^3 \) to \( \mathbb{R} \), a vector field, (i.e., a function from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \)), or meaningless.

(a) \( \text{grad}(\text{grad}(f)) \)  
(b) \( \text{Curl}(\text{grad}(f)) - \mathbf{F} \)  
(c) \( \text{Curl}(\text{Curl}(\mathbf{F})) - \mathbf{G} \)  
(d) \( \text{Curl}(\mathbf{F}) \cdot \mathbf{G} \)  
(e) \( \text{Div}(\text{Div}(\mathbf{F})) \)  
(f) \( \text{Div}(\text{Curl}(\text{grad}(f))) \)

**Solution.** The operator Curl takes a vector field in \( \mathbb{R}^3 \) as input, and produces a vector field as output; Div takes a vector field as input and produces a scalar function as output; grad takes a scalar function as input and produces a vector field as output.

If \( f \) is a scalar function, and \( \mathbf{F} \) and \( \mathbf{G} \) vector fields, then in terms of mathematical grammar, the following expressions are

(a) \( \text{grad}(\text{grad}(f)) \): meaningless – \( \text{grad}(f) \) is a vector field, and so we can’t take its gradient.

(b) \( \text{Curl}(\text{grad}(f)) - \mathbf{F} \): a vector field. More precisely, it’s the vector field \( -\mathbf{F} \) if \( f \) is a \( C^2 \) function, since \( \text{Curl}(\text{grad}(f)) = 0 \).

(c) \( \text{Curl}(\text{Curl}(\mathbf{F})) - \mathbf{G} \): a vector field. Since \( \text{Curl}(\mathbf{F}) \) is a vector field, we can take its curl, and get another vector field, from which we can subtract \( \mathbf{G} \).

(d) \( \text{Curl}(\mathbf{F}) \cdot \mathbf{G} \): a function. Since \( \text{Curl}(\mathbf{F}) \) is a vector field, its dot product with the vector field \( \mathbf{G} \) is a function.

(e) \( \text{Div}(\text{Div}(\mathbf{F})) \): meaningless. \( \text{Div}(\mathbf{F}) \) is a function, and so we can’t take its divergence.

(f) \( \text{Div}(\text{Curl}(\text{grad}(f))) \): A function, although not a very exciting one. If \( f \) is a \( C^2 \) scalar function, then \( \text{Curl}(\text{grad}(f)) = 0 \). On the other hand for any \( C^2 \) vector field \( \mathbf{F} \), \( \text{Div}(\text{Curl}(\mathbf{F})) = 0 \). In the diagram where “any two operators in a row are zero”, this is the combination of all three of them in order. As long as \( f \) is a nice function, this is as zero as zero can be.

2. Compute Div and Curl for the following vector fields:

(a) \( \mathbf{F}(x, y, z) = (x, y, z) \)  
(b) \( \mathbf{F}(x, y, z) = (y z, x z, x y) \)  
(c) \( \mathbf{F}(x, y, z) = (3 x^2 y, x^3 + y^3, z^4) \)  
(d) \( \mathbf{F}(x, y, z) = (e^x \cos(y) + z^2, e^x \sin(y) + x z, x y) \)
Solution.

(a) If $F(x, y, z) = (x, y, z)$ then $\text{Curl}(F) = (0, 0, 0)$ and $\text{Div}(F) = 3$.

(b) If $F(x, y, z) = (yz, xz, xy)$ then $\text{Curl}(F) = (0, 0, 0)$ and $\text{Div}(F) = 0$.

(c) If $F(x, y, z) = (3x^2y, x^3 + y^3, z^4)$ then $\text{Curl}(F) = (0, 0, 0)$ and $\text{Div}(F) = 6xy + 3y^2 + 4z^3$.

(d) If $F(x, y, z) = (e^x \cos(y) + z^2, e^x \sin(y) + xz, xy)$ then $\text{Curl}(F) = (0, 2z - y, z + 2e^x \sin(y))$ and $\text{Div}(F) = 2e^x \cos(x)$.

3. The vector field $\vec{r}(x, y, z) = (x, y, z)$ is commonly used in physics. Let $a, b, c \in \mathbb{R}$ be any real numbers, and set $\vec{w} = (a, b, c)$.

(a) Compute $\vec{r} \times \vec{w}$.

(b) Compute the curl of your answer from (a).

Solution.

(a) $\vec{r} \times \vec{w} = (x, y, z) \times (a, b, c) = (yc - zb, za - xc, xb - ya)$.

(b) $\text{Curl}(yc - zb, za - xc, xb - ya)$

\[
= \left( \frac{\partial}{\partial y} (xb - ya) - \frac{\partial}{\partial z} (za - xc), \frac{\partial}{\partial z} (yc - zb) - \frac{\partial}{\partial x} (xb - ya), \frac{\partial}{\partial x} (za - xc) - \frac{\partial}{\partial y} (yc - zb) \right)
\]

\[
= (-a - a, -b - b, -c - c) = (-2a, -2b, -2c) = -2(a, b, c).
\]

4.

(a) Is there a vector field $\vec{F}$ such that $\text{Curl}(\vec{F}) = (xy^2, yz^2, zx^2)$? Explain.

(b) Is there a vector field $\vec{F}$ so that $\text{Curl}(\vec{F}) = (2, 1, 3)$? If so, find one.

Solution.

(a) For $\vec{G} = (xy^2, yz^2, zx^2)$, if there were a vector field $\vec{F}$ with $\vec{G} = \text{Curl}(\vec{F})$, then we would have $\text{Div}(\vec{G}) = 0$. Since $\text{Div}(\vec{G}) = y^2 + z^2 + x^2 \neq 0$ (as functions) there is no such vector field $\vec{F}$, i.e., $\vec{G}$ is not the curl of any vector field $\vec{F}$.

(b) For $\vec{G} = (2, 1, 3)$, we do have $\text{Div}(\vec{G}) = 0$. Since $\vec{G}$ is defined on all of $\mathbb{R}^3$, and since every loop in $\mathbb{R}^3$ can be contracted to a point, the theorem from class guarantees that there is a vector field $\vec{F}$ with $\text{Curl}(\vec{F}) = \vec{G}$. Here are two methods to find one.
1. We can use the method of Q3(b). In that question we found that for any constants \((a, b, c)\), \(\text{Curl}\left((x, y, z) \times (a, b, c)\right) = -2(a, b, c)\). If we want a vector field \(F\) with \(\text{Curl}(F) = (2, 1, 3)\), we can set \((a, b, c) = (-1, -\frac{1}{2}, -\frac{3}{2})\) and then

\[
F = (x, y, z) \times (-1, -\frac{1}{2}, -\frac{3}{2}) = \left(\frac{3}{2} + \frac{z}{2}, -z + \frac{3x}{2}, -\frac{x}{2} + y\right)
\]

is a vector field so that \(\text{Curl}(F) = (3, 1, 2)\).

2. Since all the entries of \(G\) are constant, and since computing Curl involves taking first derivatives, this suggests looking for a vector field \(F\) whose entries are linear functions of \(x, y,\) and \(z\).

For a vector field \(F\) of the form \(F(x, y, z) = (ay + bz, cx + dz, ex + fy)\), we have \(\text{Curl}(F) = (f - d, b - e, c - a)\). Picking any \(a, b, c, d, e, f\) with \(f - d = 2, b - e = 1,\) and \(c - a = 3\) will give an \(F\) with \(\text{Curl}(F) = G\), for instance,

\[
F(x, y, z) = (z, 3x, 2y)
\]

will work.

5. A vector field \(F\) is called incompressible if \(\text{Div}(F) = 0\), and irrotational if \(\text{Curl}(F) = 0\).

(a) Show that any vector field of the form \(F(x, y, z) = (f(x), g(y), h(z))\) is irrotational.

(b) Show that any vector field of the form \(F(x, y, z) = (f(y, z), g(x, z), h(x, y))\) is incompressible.

(c) Find constants \(a, b,\) and \(c\) so that the vector field \(F(x, y, z) = (3x - y + az, bx - z, 4x + cy)\) is irrotational. For these values of \(a, b,\) and \(c\), find a function \(f\) with \(\nabla f = F\).

The Laplacian of a function \(f: \mathbb{R}^n \rightarrow \mathbb{R}\) is defined to be \(\text{Div}(\text{grad}(f))\), and is denoted \(\Delta f\). (NOTE: \(\Delta\) may look a bit like the gradient, \(\nabla\), but it isn’t – it’s the Laplacian.) Composing the gradient and the divergence, we compute that for a function \(f: \mathbb{R}^n \rightarrow \mathbb{R}\),

\[
\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.
\]

A function \(f\) is called harmonic if \(\Delta f = 0\).

(d) If \(F\) is a vector field defined on all of \(\mathbb{R}^3\) which is both incompressible and irrotational, show that \(F\) is the gradient of a harmonic function \(f\).
Solution.

(a) For a vector field of the form $\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$, all the partial derivatives involved in computing the curl are zero, hence the curl is zero.

(b) For a vector field of the form $\mathbf{F}(x, y, z) = (f(y, z), g(x, z), h(x, y))$, all the partial derivatives involved in computing the divergence are zero, hence the divergence is zero.

(c) If $\mathbf{F}(x, y, z) = (3x - y + az, bx - z, 4x + cy)$, then

$$\text{Curl}(\mathbf{F}) = (c + 1, a - 4, b + 1),$$

In order for this to be zero, we need $a = 4$, $b = -1$, and $c = -1$.

A function $f$ with $\text{grad}(f) = (3x - y + 4z, -x - z, 4x - y)$ is

$$f(x, y, z) = \frac{3}{2}x^2 - xy + 4xz - yz.$$

(d) If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ with $\text{Curl}(\mathbf{F}) = 0$, then we know from the theorem in class that there is a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ with $\text{grad}(f) = \mathbf{F}$. By hypothesis, $\text{Div}(\mathbf{F}) = 0$ too, so $\text{Div}(\text{grad}(f)) = 0$ since $\text{grad}(f) = \mathbf{F}$. But $\text{Div}(\text{grad}(f)) = \Delta f$ by definition of the Laplacian. Therefore $\Delta f = 0$, and so $\mathbf{F}$ is the gradient of a harmonic function.

Note: The solution to 5(d) did not involve any argument other than connecting the definitions and statements together in order. That is a good lesson to remember: something which appears complicated may actually be simple once we sort out what the statement is saying.