1. Describe the volume being integrated over, and compute the iterated integral.

\[
\int_0^5 \int_0^{\sqrt{25 - x^2}} \int_0^{\sqrt{x^2 + y^2}} z \, dz \, dy \, dx
\]

(b) \[
\int_0^{2\pi} \int_0^{1+\sin(x)} \int_0^{1+\sin(x)} 2z \, dz \, dy \, dx
\]

Solution.

(a) The region of integration is a cone. Specifically it is the cone lying below the plane \(z = 5\) and above the graph of \(z = x^2 + y^2\).

\[
\int_0^5 \int_0^{\sqrt{25 - x^2}} \int_0^{\sqrt{x^2 + y^2}} z \, dz \, dy \, dx
\]

\[
= \int_0^5 \int_0^{\sqrt{25 - x^2}} \left( \frac{1}{2} z^2 \right)_{z=\sqrt{x^2+y^2}}^{z=5} dy \, dx
\]

\[
= \frac{1}{2} \int_0^5 \int_0^{\sqrt{25 - x^2}} \left( 25 - x^2 - y^2 \right) dy \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \int_0^5 \left( 25 - r^2 \right) r \, dr \, d\theta
\]

\[
= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left( \frac{25}{2} r^2 - \frac{1}{4} r^4 \right)_{r=0}^{r=5} d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{625}{4} \, d\theta = \frac{625\pi}{16}.
\]

In the third equality above we converted to polar coordinates to make the integration simpler.

(b) The shadow of the region on the \(xy\)-plane is the shifted sine graph shown below.

The three dimensional region of integration is obtained by covering the shadow with squares parallel to the \(yz\)-plane. The base of each square lies on the \(xy\)-plane, and the size of the square is the height of the shadow at that point.
The integral is
\[
\int_0^{2\pi} \int_0^{1+\sin(x)} \int_0^{1+\sin(x)} 2z \,dz \,dy \,dx = \int_0^{2\pi} \int_0^{1+\sin(x)} \left(z^2\right)_{z=0}^{z=1+\sin(x)} \,dy \,dx
\]
\[
= \int_0^{2\pi} \int_0^{1+\sin(x)} \left(1 + \sin(x)\right)^2 \,dy \,dx = \int_0^{2\pi} \left(1 + \sin(x)\right)^2 y_{y=1+\sin(x)}^{y=0} \,dx
\]
\[
= \int_0^{2\pi} \left(1 + \sin(x)\right)^3 \,dx = \int_0^{2\pi} 1 + 3 \sin(x) + 3 \sin^2(x) + \sin^3(x) \,dx
\]
\[
= 2\pi + 3 \cdot 0 + 3 \cdot \pi + 0 = 5\pi.
\]

Here we know that the integral of \(\sin^3(x)\) over \([0, 2\pi]\) is zero by symmetry, and we have computed the integral of \(\sin^2(x)\) over \([0, 2\pi]\) in class several times. (Alternatively, use the antiderivatives \(\int \sin^2(x) \,dx = \frac{1}{2}(x - \sin(x) \cos(x))\) and \(\int \sin^3(x) \,dx = -\frac{1}{3} \sin^2(x) \cos(x) - \frac{2}{3} \cos(x)\).)

2. Sketch the region of integration for the iterated integral \(\int_0^{1} \int_x^{1} \int_0^{y-x} f \,dz \,dy \,dx\), and express it in the five other possible orders of integration.

Suggestions: (1) The region being integrated over is a tetrahedron, and it may help to work out its vertices. (2) Besides a 3D sketch, it will also be helpful to sketch the projection of the region on the \(xy\)-, \(xz\)-, and \(yz\)-planes.

Solution. The tetrahedron has vertices (0,0,0), (0,1,0), (0,1,1), and (1,1,0). Here are three views of the tetrahedron, as well as its projections onto the \(xy\)-, \(yz\)-, and \(xz\)-planes.
The bottom, right hand side, and back faces have equations (respectively) : $z = 0$, $y = 1$, and $x = 0$. The front slanted face has equation $x - y + z = 0$. The six different ways to write the iterated integral are :

$\begin{align*}
(1) & \int_{0}^{1} \int_{x}^{1} \int_{0}^{y-x} f \, dz \, dy \, dx \\
(2) & \int_{0}^{1} \int_{0}^{y} \int_{0}^{y-x} f \, dz \, dx \, dy \\
(3) & \int_{0}^{1} \int_{z}^{1} \int_{0}^{y-z} f \, dz \, dy \, dx \\
(4) & \int_{0}^{1} \int_{0}^{y} \int_{0}^{y-z} f \, dx \, dz \, dy \\
(5) & \int_{0}^{1} \int_{0}^{1-x} \int_{x+z}^{1} f \, dy \, dz \, dx \\
(6) & \int_{0}^{1} \int_{0}^{1} \int_{x+z}^{1} f \, dy \, dx \, dz
\end{align*}$

Here are some details on how these were deduced. 

**(1) and (2)** : when the inner variable is $z$, and the outer two $x$ and $y$ (in either order). The outer two integrals parameterize the shadow of the tetrahedron on the $xy$-plane. To work out the innermost $z$-limits, we imagine a point in shadow on the $xy$-plane, and ask, as we go up, when to start and when to stop integrating. We start when we cross the bottom plane $(z = 0)$, and stop when we cross the front slanted face $(z = y - x)$.

**(3) and (4)** : when the inner variable is $x$, and the outer two $y$ and $z$ (in either order). The outer two integrals parameterize the shadow of the tetrahedron on the $yz$-plane. To work out the innermost $x$-limits, we imagine a point in shadow on the $yz$-plane, and ask, as we go from back to front, when to start and when to stop integrating. We start when we cross the bottom plane $(x = 0)$, and stop when we cross the front slanted face $(x = y - z)$. 


(5) and (6) : when the inner variable is \( y \), and the outer two \( x \) and \( z \) (in either order). The outer two integrals parameterize the shadow of the tetrahedron on the \( xz \)-plane. To work out the innermost \( y \)-limits, we imagine a point in shadow on the \( xz \)-plane, and ask, as we go from left to right, when to start and when to stop integrating. We start when we cross the front slanted face \( (y = x + z) \), and stop when we cross the rightmost face \( (y = 1) \).

3. For each of the following surfaces, find a parameterization, and compute the tangent vectors and normal vectors in terms of that parameterization:

(a) The graph of \( f(x, y) = 9 - xy \) over the circle \( x^2 + y^2 \leq 9 \).

(b) The part of the sphere \( x^2 + y^2 + z^2 = 4 \) above the plane \( z = 1 \) (i.e., with \( z \geq 1 \)).

(c) The surface obtained by rotating the line segment connecting \((1, 0, 2)\) to \((4, 0, 0)\) about the \( z \)-axis.

Solution.

(a) The surface is the graph of a function over a region in \( \mathbb{R}^2 \), and we have a general strategy to deal with this situation: first parametrize the region and then plug the parameterization into the function to determine the \( z \)-coordinate. In this case the region \( R \subset \mathbb{R}^2 \) is the disc of radius 3.

One possible parameterization of \( R \) is by using polar coordinates. Then the parametrization becomes

\[
\begin{align*}
x(r, \theta) &= r \cos(\theta) \\
y(r, \theta) &= r \sin(\theta) \\
z(r, \theta) &= 9 - r^2 \cos(\theta) \sin(\theta) \\
(r, \theta) &\in [0, 3] \times [0, 2\pi]
\end{align*}
\]

Using the sine and cosine addition formulae, the tangent vectors can also be written as:

\[
\begin{align*}
T_r &= (\cos(\theta), \sin(\theta), -2r \cos(\theta) \sin(\theta)) \\
T_\theta &= (-r \sin(\theta), r \cos(\theta), -r^2 \cos(\theta) + r^2 \sin^2(\theta)) \\
N &= (r^2 \sin(\theta), r^2 \cos(\theta), r)
\end{align*}
\]
(b) For parameterizing the sphere, or part of the sphere, it is often easiest to use spherical coordinates. In this case, since the sphere has radius 2, the condition that \( z \geq 1 \) amounts to the condition that \( z(\theta, \varphi) = 2 \sin(\varphi) \geq 1 \), or \( \sin(\varphi) \geq \frac{1}{2} \), or \( \varphi \geq \frac{\pi}{6} \).

\[
\begin{align*}
  x(\theta, \varphi) &= 2 \cos(\theta) \cos(\varphi) \quad T_\theta = (-2 \sin(\theta) \cos(\varphi), 2 \cos(\theta) \cos(\varphi), 0) \\
  y(\theta, \varphi) &= 2 \sin(\theta) \cos(\varphi) \quad T_\varphi = (-2 \cos(\theta) \sin(\varphi), -2 \sin(\theta) \sin(\varphi), 2 \cos(\varphi)) \\
  z(\theta, \varphi) &= 2 \sin(\varphi) \quad N = (4 \cos(\theta) \cos^2(\varphi), 4 \sin(\theta) \cos^2(\varphi), 4 \sin(\varphi) \cos(\varphi)) .
\end{align*}
\]

\((\theta, \varphi) \in [0, 2\pi] \times [\frac{\pi}{6}, \frac{\pi}{2}]\)

(c) The segment is a piece of the line \( z = -\frac{2}{3}x + \frac{8}{3}, y = 0 \), which is contained in the \( xz \)-plane. Rotated around the \( z \)-axis the surface obtained is a piece of a cone.

The projection of this surface onto the \( xy \)-plane is an annulus, with inner radius 1 and outer radius 4. Over this region \( R \), the surface is a graph of a function, namely the function \( z = -\frac{2}{3}r + \frac{8}{3} \), where \( r \) is the distance to the origin. Again it is easiest to parametrize the projection \( R \) using polar coordinates. This gives the parametrization

\[
\begin{align*}
  x(r, \theta) &= r \cos(\theta) \quad T_r = (\cos(\theta), \sin(\theta), -\frac{2}{3}) \\
  y(r, \theta) &= r \sin(\theta) \quad T_\theta = (-r \sin(\theta), r \cos(\theta), 0) \\
  z(r, \theta) &= -\frac{2}{3}r + \frac{8}{3} \quad N = (\frac{2}{3}r \cos(\theta), \frac{2}{3}r \sin(\theta), r)
\end{align*}
\]

4. Let \( S \) be the helicoid parameterized by \((v \cos(\theta), v \sin(\theta), \theta)\) with \((\theta, v) \in [0, 4\pi] \times [0, 1]\). A picture of the helicoid is shown at right.

(a) Find the area of \( S \) (equivalently, find \( \iint_S 1 \, dS \)).

(b) Find the average value of the function \( f(x, y, z) = yz \) over \( S \).

The antiderivative \( \int \sqrt{1 + u^2} \, du = \frac{1}{2} \left( u \sqrt{1 + u^2} + \ln \left( u + \sqrt{1 + u^2} \right) \right) \) may be useful in this question.

**Solution.** In order to carry out the integration, we will need to work out the formula for the normal vector. We have

\[
\begin{align*}
  T_\theta &= (-v \sin(\theta), v \cos(\theta), 1), \\
  T_v &= (\cos(\theta), \sin(\theta), 0), \\
  N &= (-\sin(\theta), \cos(\theta), -v),
\end{align*}
\]

with \( |N| = \sqrt{(-\sin(\theta))^2 + (\cos(\theta))^2 + (-v)^2} = \sqrt{1 + v^2} \).
(a) By our method of computing the integral of a function over a surface,

\[
\int_{S} 1 \, dS = \int_{0}^{4\pi} \int_{0}^{1} 1 \cdot \|\mathbf{N}\| \, dv \, d\theta = \int_{0}^{4\pi} \int_{0}^{1} \sqrt{1 + v^2} \, dv \, d\theta
\]

\[
= \frac{1}{2} \left( v \sqrt{1 + v^2} + \ln \left( v + \sqrt{1 + v^2} \right) \right)_{v=0}^{v=1} d\theta = \frac{1}{2} \int_{0}^{4\pi} \sqrt{2} + \ln(1 + \sqrt{2}) \, d\theta
\]

\[
= 2\pi \left( \sqrt{2} + \ln(1 + \sqrt{2}) \right).
\]

(b) Similarly,

\[
\int_{S} yz \, dS = \int_{0}^{4\pi} \int_{0}^{1} (v \sin(\theta)) \cdot \theta \cdot \|\mathbf{N}\| \, dv \, d\theta = \int_{0}^{4\pi} \int_{0}^{1} (v \sin(\theta)) \cdot \theta \cdot \sqrt{1 + v^2} \, dv \, d\theta
\]

\[
= \frac{1}{3} (2^3 - 1) \int_{0}^{4\pi} \theta \sin(\theta) \, d\theta = \frac{1}{3} (2^3 - 1) \left( \sin(\theta) - \theta \cdot \cos(\theta) \right)_{\theta=0}^{\theta=2\pi}
\]

\[
= -\frac{4\pi}{3} (2^3 - 1).
\]

The average value of the function \( yz \) over the surface \( S \) is the ratio of these two values (i.e., the integral of the function divided by the area of the surface)

\[
\frac{-\frac{4\pi}{3} (2^3 - 1)}{2\pi \left( \sqrt{2} + \ln(1 + \sqrt{2}) \right)} = \frac{-2 (2^3 - 1)}{3 \left( \sqrt{2} + \ln(1 + \sqrt{2}) \right)}.
\]

5. Fix a radius \( r > 0 \) and two angles \( \varphi_1 \) and \( \varphi_2 \), with \(-\frac{\pi}{2} \leq \varphi_1 \leq \varphi_2 \leq \frac{\pi}{2}\). Find the surface area of the portion of the sphere of radius \( r \) with latitudes between \( \varphi_1 \) and \( \varphi_2 \).

**Solution.** We can use the usual parameterization for a sphere of radius \( r \) by longitude \( (\theta) \) and latitude \( (\varphi) \):

\[
x(u, v) = r \cos(\theta) \cos(\varphi) \quad T_{\theta} = (-r \sin(\theta) \cos(\varphi), r \cos(\theta) \cos(\varphi), 0)
\]
\[
y(u, v) = r \sin(\theta) \cos(\varphi) \quad T_{\varphi} = (-r \cos(\theta) \sin(\varphi), -r \sin(\theta) \sin(\varphi), r \cos(\varphi))
\]
\[
z(u, v) = r \sin(\varphi) \quad \mathbf{N} = (r^2 \cos(\theta) \cos^2(\varphi), r^2 \sin(\theta) \sin^2(\varphi), r^2 \sin(\varphi) \cos(\varphi))
\]

with \( \varphi_1 \leq \varphi \leq \varphi_2 \), \( 0 \leq \theta \leq 2\pi \).

The length of the normal vector is \( \|\mathbf{N}\| = r^2 \cos(\varphi) \). To find the area, we integrate the function \( f = 1 \), so if \( S \) is the portion of the sphere between angles \( \varphi_1 \) and \( \varphi_2 \), we need to compute

\[\]
\begin{align*}
\int \int_S 1 \, dS &= \int_0^{2\pi} \int_{\varphi_1}^{\varphi_2} r^2 \cos(\varphi) \, d\varphi \, d\theta \\
&= \int_0^{2\pi} r^2 (\sin(\varphi_2) - \sin(\varphi_1)) \, d\theta = 2\pi r^2 (\sin(\varphi_2) - \sin(\varphi_1)).
\end{align*}

There is a nice solution to this problem which has been known for over two thousand years. It is a wonderful theorem of Archimedes (287 B.C. – 212 B.C.) that if you surround a sphere with a cylinder of the same radius and height:

Then projection “sideways” from the vertical axis of the sphere onto the surface of the cylinder is an area preserving projection. I.e., whatever shape you draw on the surface of the sphere, when you project it onto the cylinder it may end up distorted, but it will still have the same area.

If we project the region of the problem onto the cylinder, it becomes a horizontal strip of the cylinder, of height \( r(\sin(\varphi_2) - \sin(\varphi_1)) \). Since the cylinder has radius \( r \), this is of area \( (2\pi r)(r(\sin(\varphi_2) - \sin(\varphi_1))) = 2\pi r^2 (\sin(\varphi_2) - \sin(\varphi_1)) \), just as we computed above.