1. Find the integral $\iint_S \mathbf{F} \cdot dS$ where $S$ is the helicoid parameterized by $(u \cos(v), u \sin(v), v)$, $0 \leq u \leq 3$, $0 \leq v \leq 4\pi$ with positive orientation upwards, and where $\mathbf{F}$ is the vector field $\mathbf{F}(x, y, z) = (xz, -yz, xy)$.

Solution. We have

\[
\begin{align*}
\mathbf{T}_u &= (\cos(v), \sin(v), 0), \\
\mathbf{T}_v &= (-u \sin(v), u \cos(v), 1), \\
\mathbf{N} &= \mathbf{T}_u \times \mathbf{T}_v = \left( \sin(v), -\cos(v), u \right), \\
\text{and} \\
\mathbf{F} &= \left( uv \cos(v), -uv \sin(v), u^2 \cos(v) \sin(v) \right).
\end{align*}
\]

The orientation matches the orientation of $S$. We compute that

\[
\begin{align*}
\mathbf{F} \cdot \mathbf{N} &= \left( uv \cos(v), -uv \sin(v), u^2 \cos(v) \sin(v) \right) \cdot \left( \sin(v), -\cos(v), u \right) \\
&= 2uv \cos(v) \sin(v) + u^3 \cos(v) \sin(v) \\
&= uv \sin(2v) + \frac{u^3}{2} \sin(2v).
\end{align*}
\]

Therefore

\[
\begin{align*}
\iint_S \mathbf{F} \cdot dS &= \int_0^3 \int_0^{4\pi} uv \sin(2v) + \frac{u^3}{2} \sin(2v) \ dv \ du \\
&= \int_0^3 \left[ \frac{u}{2} \sin(2v) - \frac{u}{2} v \cos(2v) - \frac{u^3}{4} \cos(2v) \right]_{v=0}^{v=4\pi} \ du \\
&= \int_0^3 -2u \pi \ du = \left( -\pi u^2 \right)_{u=0}^{u=3} = -9\pi.
\end{align*}
\]

2. Find the flux integral of $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$ through the top half of the unit sphere, with outward orientation.

Solution. We use the standard parameterization of the sphere via latitude and longitude:

\[
\begin{align*}
x(u, v) &= \cos(\theta) \cos(\varphi) \\
y(u, v) &= \sin(\theta) \cos(\varphi) \\
z(u, v) &= \sin(\varphi) \\
(\theta, \varphi) &\in [0, 2\pi] \times [0, \frac{\pi}{2}].
\end{align*}
\]

The orientation above is outwards, i.e., agrees with the orientation we have chosen. The restriction $\varphi \in [0, \frac{\pi}{2}]$ (rather than $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$) ensures that we are only parameterizing the top of the sphere.
In terms of the parameterization, we have

\[ \mathbf{F} = \left( \cos^2(\theta) \cos^2(\varphi), \sin^2(\theta) \cos^2(\varphi), \sin^2(\varphi) \right) \]

and

\[ \mathbf{F} \cdot \mathbf{N} = \left( \cos^2(\theta) \cos^2(\varphi), \sin^2(\theta) \cos^2(\varphi), \sin^2(\varphi) \right) \cdot \left( \cos(\theta) \cos^2(\varphi), \sin(\theta) \cos^2(\varphi), \sin(\varphi) \cos(\varphi) \right) = \cos^3(\theta) \cos^4(\varphi) + \sin^3(\theta) \cos^4(\varphi) + \sin^3(\varphi) \cos(\varphi), \]

so that

\[
\int_{S} \mathbf{F} \cdot dS = \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{2}} \cos^3(\theta) \cos^4(\varphi) + \sin^3(\theta) \cos^4(\varphi) + \sin^3(\varphi) \cos(\varphi) \, d\varphi \, d\theta.
\]

By reasons of symmetry the integrals of \(\cos^3(\theta)\) and \(\sin^3(\theta)\) over \(\theta \in [0, 2\pi]\) are zero. Alternatively, we can calculate this directly.

\[
\int_{0}^{2\pi} \cos^3(\theta) \, d\theta = \int_{0}^{2\pi} \cos(\theta) \left( 1 - \sin^2(\theta) \right) \, d\theta = \int_{0}^{2\pi} \cos(\theta) - \cos(\theta) \sin^2(\theta) \, d\theta = \left( \sin(\theta) - \frac{1}{3} \sin^3(\theta) \right)_{\theta=2\pi}^{\theta=0} = 0.
\]

and

\[
\int_{0}^{2\pi} \sin^3(\theta) \, d\theta = \int_{0}^{2\pi} \sin(\theta) \left( 1 - \cos^2(\theta) \right) \, d\theta = \int_{0}^{2\pi} \sin(\theta) - \sin(\theta) \cos^2(\theta) \, d\theta = \left( -\cos(\theta) + \frac{1}{3} \cos^3(\theta) \right)_{\theta=2\pi}^{\theta=0} = 0.
\]

In our parameterization we are integrating over a \((\theta, \varphi)\) rectangle, so Fubini’s theorem tells us that we can integrate in either order. Switching the order, and using the above result about the integral of \(\cos^3(\theta)\) and \(\sin^3(\theta)\), the surface integral becomes

\[
\int_{S} \mathbf{F} \cdot dS = \int_{0}^{\frac{\pi}{2}} \int_{0}^{2\pi} \cos^3(\theta) \cos^4(\varphi) + \sin^3(\theta) \cos^4(\varphi) + \sin^3(\varphi) \cos(\varphi) \, d\theta \, d\varphi = \int_{0}^{\frac{\pi}{2}} \sin^3(\varphi) \cos(\varphi) \, d\varphi = 2\pi \left( \frac{1}{4} \sin^4(\theta) \right)_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{\pi}{2}.
\]
3. The parameterized curve \( c(t) = (5 \cos(t) + \sin(5t), 5 \sin(t) + \cos(5t)) \)
for \( t \in [0, 2\pi] \) is shown at right. Use the vector field \( F = \frac{1}{2}(-y, x) \)
and Green’s theorem to find the area enclosed by the curve.

The angle addition formula \( \sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta) \)
may prove useful at some point in the calculation.

**Solution.** Let \( R \) be the region enclosed by the curve. The parameterization \( c \) is oriented
counterclockwise. Since \( \text{Curl}_2(F) = \frac{1}{2} \left( \frac{\partial}{\partial x} x - \frac{\partial}{\partial y} (-y) \right) = 1 \),
by Green’s theorem we have

\[
\int_c F \cdot ds = \iint_R \text{Curl}_2(F) \, dA = \iint_R 1 \, dA = \text{Area}(R).
\]

In other words, we can compute the area of \( R \) by computing the integral of \( F \) around \( c \).

We have

\[
c'(t) = (-5 \sin(t) + 5 \cos(5t), 5 \cos(t) - 5 \sin(5t)) = 5 \left( -\sin(t) + \cos(5t), \cos(t) - \sin(5t) \right)
\]

and

\[
F(c(t)) = \frac{1}{2} \left( -5 \sin(t) - \cos(5t), 5 \cos(t) + \sin(5t) \right),
\]

so that

\[
F(c(t)) \cdot c'(t) = 5 \cdot \frac{1}{2} \left( 5 \sin^2(t) + \sin(t) \cos(5t) - 5 \sin(t) \cos(5t) - \cos^2(5t)
+ 5 \cos^2(t) + \cos(t) \sin(5t) - 5 \cos(t) \sin(5t) - \sin^2(5t) \right)
\]

\[
= \frac{5}{2} \left( 5 - 1 - 4 (\sin(t) \cos(5t) + \cos(t) \sin(5t)) \right)
\]

\[
= 10 - 10 \left( \sin(t) \cos(5t) + \cos(t) \sin(5t) \right) = 10 - 10 \sin(6t).
\]

Therefore

\[
\text{Area}(R) = \int_c F \cdot ds = \int_0^{2\pi} 10 - 10 \sin(6t) \, dt = \left( 10t + \frac{5}{3} \cos(6t) \right)_{t=0}^{t=2\pi} = 20\pi.
\]

4. Compute the following line integrals by using Green’s Theorem to convert each of
them into an integral over a two-dimensional region \( R \), and then evaluating that integral
over \( R \).

(a) Compute \( \int_c F \cdot ds \), where \( c \) is the circle of radius 2, centered at \((0,0)\), oriented
counterclockwise, and \( F(x,y) = \left( \cos(\cos(x)) - x^2 y, e^{\sin(y^2)} + x y^2 \right) \).

(b) Compute \( \int_c F \cdot ds \), where \( c \) is the boundary of \([1,2] \times [-1,1]\), oriented counter-
clockwise, and \( F(x,y) = \left( x y^2 + x^3, e^{x^2} + e^{y^2} \right) \).
(c) Compute \( \int_c F \cdot ds \), where \( c \) is the boundary of the region between \( y = x^2 - 4x \) and \( y = 5 \), oriented counterclockwise, and \( F(x, y) = (y, x^2y) \).

Solution.

(a) \( \text{Curl}_2(F) = \frac{\partial}{\partial x} \left( e^{\sin(y^2)} + xy^2 \right) - \frac{\partial}{\partial y} \left( \cos(\cos(x)) - x^2y \right) = y^2 + x^2 \). Letting \( R \) be the disc of radius 2 centered at the origin (i.e., the disc whose boundary is \( c \)), then by Green’s theorem we have

\[
\int_c F \cdot ds = \iint_R x^2 + y^2 \, dA.
\]

Converting to polar coordinates (i.e., \( x(r, \theta) = r \cos(\theta), \quad y(r, \theta) = r \sin(\theta) \)), \( R \) is the polar rectangle \( (r, \theta) \in [0, 2] \times [0, 2\pi] \), and \( x^2 + y^2 = r^2 \). The change of variables theorem tells us that the area distortion factor for polar coordinates is

\[
\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \left| \begin{array}{cc} \cos(\theta) & \sin(\theta) \\ -r \sin(\theta) & r \cos(\theta) \end{array} \right| = r.
\]

Therefore,

\[
\int_c F \cdot ds = \iint_R x^2 + y^2 \, dA = \int_0^{2\pi} \int_0^2 r^2 \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| \, dr \, d\theta
\]

\[
= \int_0^2 \int_0^{2\pi} r^3 \, d\theta \, dr \int_0^2 2\pi r^3 \, dr = \frac{\pi}{2} \left( r^4 \right)^2_{r=0} = 8\pi.
\]

(b) \( \text{Curl}_2(F) = \frac{\partial}{\partial x} \left( e^{x^2} + e^{y^2} \right) - \frac{\partial}{\partial y} \left( xy^2 + x^3 \right) = 2xe^{x^2} - 2xy \). Let \( R = [1, 2] \times [-1, 1] \) then Green’s theorem gives us

\[
\int_c F \cdot ds = \iint_R 2xe^{x^2} - 2xy \, dA = \int_1^2 \int_{-1}^1 \left( 2xe^{x^2} - 2xy \right) \, dy \, dx
\]

\[
= \int_1^2 \left( 2xye^{x^2} - xy^2 \right)_{y=-1}^{y=1} \, dx = \int_1^2 4xe^{x^2} \, dx = \left( 2e^{x^2} \right)_{x=1}^{x=2} = 2e^4 - 2e.
\]
(c) \( \text{Curl}_2(\mathbf{F}) = \frac{\partial}{\partial x} (x^2y) - \frac{\partial}{\partial y} (y) = 2xy - 1 \). Let \( R \) be the region between \( y = x^2 - 4x \) and \( y = 5 \). By Green’s theorem we have

\[
\int_{c} \mathbf{F} \cdot d\mathbf{s} = \iint_{R} \left( 2xy - 1 \right) \, dA
\]

\[
= \int_{-1}^{5} \int_{x^2-4x}^{5} \left( 2xy - 1 \right) \, dy \, dx
\]

\[
= \int_{-1}^{5} \left( xy^2 - y \right)_{x^2-4x}^{5} \, dx
\]

\[
= \int_{-1}^{5} \left( x(5^2 - (x^2 - 4x)^2) - (5 - (x^2 - 4x)) \right) \, dx
\]

\[
= \left. \left( -\frac{1}{6}x^6 + \frac{8}{5}x^5 - 4x^4 + \frac{1}{3}x^3 + \frac{21}{2}x^2 - 5x \right) \right|_{x=-1}^{x=5}
\]

\[
= \frac{828}{5}.
\]

5. Consider the following integral, which does not seem very easy to evaluate.

\[
(*) \quad \frac{1}{\pi} \int_{0}^{2\pi} e^{100\cos^2(t)} \sin(1 + e^{30\cos^2(t)}) \sin(t) + \cos^2(t) \, dt
\]

In this problem we will evaluate the integral by using Green’s theorem. Let \( c \) be the circle of radius 1 centered at \((0,0)\), and oriented counterclockwise. One possible parameterization of \( c \) is \( c(t) = (\cos(t), \sin(t)) \) with \( t \in [0, 2\pi] \).

(a) Find a vector field \( \mathbf{F} \) so that when evaluating \( \int_{c} \mathbf{F} \cdot d\mathbf{s} \) using the parameterization above, the integral that results is \((*)\).

(b) Use Green’s theorem to convert this to an integral over the unit disc, and evaluate that integral.

Solution.

(a) In the given parameterization we have \( c'(t) = (-\sin(t), \cos(t)) \). When computing \( \int_{c} \mathbf{F} \cdot d\mathbf{s} \) we will be taking the dot product of \( \mathbf{F}(c(t)) \) and \( c'(t) \). If the components of \( \mathbf{F} \) are \( \mathbf{F} = (F_1, F_2) \), then this dot product is \( F_1(c(t))(-\sin(t)) + F_2(c(t))(\cos(t)) \), so to guess at \( \mathbf{F} \) it would be a good idea to write the integrand as a term multiplied by \(-\sin(t)\) added to a term multiplied by \(\cos(t)\). One obvious way to do this is to write
\[\frac{1}{\pi} \left( e^{100 \cos^2(t)} \sin(1 + e^{30 \cos^2(t)}) \sin(t) + \cos^2(t) \right)\]

\[= \frac{1}{\pi} \left( -e^{100 \cos^2(t)} \sin(1 + e^{30 \cos^2(t)}) \right) (-\sin(t)) + \frac{1}{\pi} (\cos(t)) (\cos(t)).\]

This suggests that we want

\[F_1(c(t)) = \frac{1}{\pi} \left( -e^{100 \cos^2(t)} \sin(1 + e^{30 \cos^2(t)}) \right)\]

and

\[F_2(c(t)) = \frac{1}{\pi} \cos(t).\]

The next step is to see if we can choose \(F_1\) and \(F_2\), functions of \(x\) and \(y\), so that when we compute \(F_1(c(t))\) and \(F_2(c(t))\) (i.e., when we substitute \(x = \cos(t)\) and \(y = \sin(t)\)), we get the expressions above. One solution is

\[F_1(x, y) = \frac{1}{\pi} \left( -e^{100 x^2} \sin(1 + e^{30 x^2}) \right)\]

and \(F_2(x, y) = \frac{1}{\pi} x\),

so that

\[F(x, y) = \frac{1}{\pi} \left( -e^{100 x^2} \sin(1 + e^{30 x^2}), x \right).\]

(b) We have \(\text{Curl}_2(\mathbf{F}) = \frac{1}{\pi} \left( \frac{\partial}{\partial x} (x) - \frac{\partial}{\partial y} \left( -e^{100 x^2} \sin(1 + e^{30 x^2}) \right) \right) = \frac{1}{\pi}\). Setting \(R\) to be the unit disk (i.e., the disk of radius 1 centered at \((0, 0)\), Green’s theorem and our choice of \(\mathbf{F}\) give us that

\[\frac{1}{\pi} \int_0^{2\pi} e^{100 \cos^2(t)} \sin(1 + e^{30 \cos^2(t)}) \sin(t) + \cos^2(t) \, dt = \int_{C} \mathbf{F} \cdot ds = \int_{R} \mathbf{F} \cdot \mathbf{n} \, dA\]

\[= \frac{1}{\pi} \cdot \text{Area}(R) = \frac{1}{\pi} \cdot \pi = 1.\]