1. Let $c$ be the top half of the unit circle, oriented from $(-1, 0)$ to $(1, 0)$, and $F$ be the vector field
\[ F(x, y) = (\ln(x + 5) + y^2, 2xy - y^2). \]
Check that $\text{Curl}_2(F) = 0$ and use the flexibility theorem for curves to compute the integral $\int_c F \cdot ds$. Flexing the curve to the line joining $(-1, 0)$ and $(1, 0)$ is a possibility.

**Solution.** For $F(x, y) = (\ln(x + 5) + y^2, 2xy - y^2)$ we have
\[
\text{Curl}_2(F) = \frac{\partial}{\partial x} (2xy - y^2) - \frac{\partial}{\partial y} (\ln(x + 5) + y^2) = 2y - 2y = 0.
\]
If $c_1$ is the top half of the unit circle oriented from $(-1, 0)$ to $(1, 0)$, it has the same endpoints as the line segment $c_2$ going from $(-1, 0)$ to $(1, 0)$.

By the flexibility theorem for curves, this means that $\int_{c_1} F \cdot ds = \int_{c_2} F \cdot ds$. Computing the integral $\int_{c_2} F \cdot ds$ is easy:
\[
\begin{align*}
x(t) &= t \\
y(t) &= 0 \\
t &\in [-1, 1].
\end{align*}
\]
Since $F(c_2(t)) \cdot c'_2(t) = \ln(t + 5)$ we have
\[
\int_{c_2} F \cdot ds = \int_{-1}^{1} \ln(t + 5) \, dt = \left( (t + 5) \ln(t + 5) - t \right)_{t=-1}^{t=1} = 6 \ln(6) - 4 \ln(4) - 2.
\]

2. Let $F$ be the vector field
\[ F(x, y, z) = (ye^{x^2} - xe^{xy}, ye^{xy} + \tan(z^2 + z + 1), x^2). \]
Check that $\text{Div}(F) = 0$, and use the flexibility theorem for surfaces to compute $\iint_S F \cdot dS$ where $S$ is the top half of the unit sphere oriented outwards. (“Flexing” it to the unit disk seems like a good bet.)

**Solution.** For $F(x, y, z) = (ye^{x^2} - xe^{xy}, ye^{xy} + \tan(z^2 + z + 1), x^2)$, we have
\[
\text{Div}(F) = \frac{\partial}{\partial x} (ye^{x^2} - xe^{xy}) + \frac{\partial}{\partial y} (ye^{xy} + \tan(z^2 + z + 1)) + \frac{\partial}{\partial z} (x^2)
\]
\[
= -e^{xy} - xe^{xy} + e^{xy} + xye^{xy} + 0 = 0.
\]
Let $S_1$ be the top half of the unit sphere (oriented upwards) and $S_2$ the unit disk in the $xy$-plane, also oriented upwards. The flexibility theorem for surfaces tells us that
\[ \int S_1 F \cdot dS = \int S_2 F \cdot dS. \]
Let’s do the integral over $S_2$, since it seems easier.

\[
\begin{align*}
x(r, \theta) &= r \cos(\theta) \quad &T_r &= (\cos(\theta), \sin(\theta), 0) \\
y(r, \theta) &= r \sin(\theta) \quad &T_\theta &= (-r \sin(\theta), r \cos(\theta), 0) \\
x(r, \theta) &= 0 &N &= (0, 0, r). \\
(r, \theta) &\in [0, 1] \times [0, 2\pi]
\end{align*}
\]

The dot product of the vector field with the normal vector is $F \cdot N = r^3 \cos(\theta)$, so the flux integral is
\[ \int S_2 F \cdot dS = \int_0^1 \int_0^{2\pi} r^3 \cos^2(\theta) \, d\theta \, dr = \frac{\pi}{4}. \]

3. Let $c_1$ be the top half of the unit circle, oriented counterclockwise, $c_2$ be the line segment joining $(1, 0)$ to $(-1, 0)$, oriented from $(1, 0)$ to $(-1, 0)$, and let $F$ be the vector field
\[ F(x, y) = (x^2 - y, xy - \arcsin(y) + e^{y^3}). \]

(a) Use Green’s theorem to compute the difference between $\int_{c_1} F \cdot ds$ and $\int_{c_2} F \cdot ds$.

(b) Use (a) to compute $\int_{c_1} F \cdot ds$ (by computing the easier $\int_{c_2} F \cdot ds$, of course...).

**Solution.**

(a) If we start with $F(x, y) = (x^2 - y, xy - \arcsin(y) + e^{y^3})$ then
\[ \text{Curl}_2(F) = \frac{\partial}{\partial x} (xy - \arcsin(y) + e^{y^3}) - \frac{\partial}{\partial y} (x^2 - y) = y + 1. \]

Let $R$ be the top half of the unit disk. With the orientations of $c_1$ and $c_2$ as stated in the problem, Green’s theorem tells us that
\[ \int_{c_1} F \cdot ds - \int_{c_2} F \cdot ds = \int \int_R \text{Curl}_2(F) \, dA = \int \int_R (y + 1) \, dA. \]

In polar coordinates, the integral of $y + 1$ over $R$ is
\[
\begin{align*}
\int \int_R (y + 1) \, dA &= \int_0^1 \int_0^{\pi} r(r \sin(\theta) + 1) \, d\theta \, dr \cdot (r \cos(\theta) + r \theta)_{\theta=\pi}^{\theta=0} \, dr \\
&= \int_0^1 2r^2 + \pi r \, dr = \left( \frac{2}{3} r^3 + \frac{1}{2} \pi r^2 \right)_{r=0}^{r=1} = \frac{2}{3} + \frac{\pi}{2}.
\end{align*}
\]
Therefore we have computed that
\[ \int_{c_1} \mathbf{F} \cdot ds - \int_{c_2} \mathbf{F} \cdot ds = \frac{2}{3} + \frac{\pi}{2}. \]

(b) We will now compute \( \int_{c_1} \mathbf{F} \cdot ds \) by using the result from part (a) and computing \( \int_{c_2} \mathbf{F} \cdot ds \) instead. We can parametrize \( c_2 \) by

\[
\begin{align*}
x(t) &= -t, & \mathbf{c}'_2(t) &= (-1, 0), \\
y(t) &= 0, & \mathbf{F}(\mathbf{c}_2(t)) &= (t^2, 1).
\end{align*}
\]

The dot product \( \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}'_2(t) \) is equal to \( t^2 \), and so

\[
\int_{c_2} \mathbf{F} \cdot ds = \int_{-1}^{1} -t^2 dt = \left(-\frac{1}{3}t^3\right)_{t=1}^{t=-1} = -\frac{2}{3}.
\]

Combining this calculation with part (a) we have Therefore we have

\[
\int_{c_1} \mathbf{F} \cdot ds = \iint_R \text{Curl}_2(\mathbf{F}) \, dA + \int_{c_2} \mathbf{F} \cdot ds = \left(\frac{2}{3} + \frac{\pi}{2}\right) + \frac{2}{3} = \frac{\pi}{2}.
\]

4. Questions about differential forms.

(a) Let \( f = \sin(xy) - 3xy^2z \). Compute the 1-form \( df \).

It turns out that doing \( d \) twice in a row always results in 0. The reason is a combination of “mixed partials commute”, and the rules for differentials (“swapping any two \( dx_i \), \( dx_j \) changes the sign”, and “any repeated \( dx_i \) means that the form is zero”).

(b) Let \( \alpha \) be your answer from (a). Check the claim above by computing the 2-form \( d\alpha \). I.e., compute \( d\alpha \) and check that it is zero, showing the details.

The \( d \) operators fit together to give a diagram

\[
\begin{align*}
\{ \text{Functions} \} \xrightarrow{d} \{ 1\text{-forms on } \mathbb{R}^3 \} & \xrightarrow{d} \{ 2\text{-forms on } \mathbb{R}^3 \} & \xrightarrow{d} \{ 3\text{-forms on } \mathbb{R}^3 \}.
\end{align*}
\]

where any two in a row is zero.

You have a friend who is does not like differential forms. Fortunately you have a great idea on how to help them. You think : “A 1-form is something that looks like
\[ F_1 \, dx + F_2 \, dy + F_3 \, dz, \] where \( F_1, F_2, \) and \( F_3 \) are functions on \( \mathbb{R}^3 \). Another way to keep track of three different functions is a vector field \( \mathbf{F} = (F_1, F_2, F_3) \) with three different components. I’ll tell my friend to forget about 1-forms, and explain everything using vector fields.”

For instance, given a function \( f \), you know that \( df = f_x \, dx + f_y \, dy + f_z \, dz \). In terms of your (vector field) \( \leftrightarrow \) (1-form) dictionary, this is the vector field \((f_x, f_y, f_z)\). You then tell your friend : “Don’t worry about the \( d \) operator that takes functions to 1-forms. If you have a function \( f \), \( d \) of that is just the vector field \((f_x, f_y, f_z)\).”

Encouraged by your success with 1-forms, you decide to simplify 2-forms and 3-forms too. A 2-form is also given by three functions, the entries in front of \( dx \wedge dy \), \( dy \wedge dz \), and \( dx \wedge dz \). You decide to match this up with a vector field by declaring that the vector field \( \mathbf{G} = (G_1, G_2, G_3) \) corresponds to the 2-form \( G_1 \, dy \wedge dz - G_2 \, dx \wedge dz + G_3 \, dx \wedge dy \). [This choice of signs will make more sense after (c) and (d) below.] You also realize that 3-forms are just multiples of \( dx \wedge dy \wedge dz \), and so can be described by a single function (e.g., the 3-form \( H \, dx \wedge dy \wedge dz \) would correspond to the function \( H \)).

(c) Using your dictionary, tell your friend how the \( d \) operator takes 2-forms to 3-forms.  
I.e., starting with \( \mathbf{G} = (G_1, G_2, G_3) \) convert \( \mathbf{G} \) to a 2-form by the rule above, apply \( d \) to that to get a 3-form, and then convert the 3-form back to a function (showing the details of your calculations). What function (in terms of \( G_1, G_2, \) and \( G_3 \)) do you get?

(d) Finally, your toughest challenge : explain to your friend how to take \( d \) of a 1-form using your dictionary. I.e., start with a vector field \( \mathbf{F} = (F_1, F_2, F_3) \), convert it to a 1-form, take \( d \) of that 1-form to get a 2-form, and then convert the 2-form back to a vector field \( \mathbf{G} \) (all using the rules above). Starting with \( \mathbf{F} \), what is the formula for the vector field \( \mathbf{G} \) that results?

Solution.

(a) For any function \( f \) on \( \mathbb{R}^3 \), \( df = f_x \, dx + f_y \, dy + f_z \, dz \). With \( f = \sin(xy) - 3xy^2z \), this gives 
\[
 df = (y \cos(xy) - 3y^2z) \, dx + (x \cos(xy) - 6xyz) \, dy + (-3xy^2) \, dz.
\]

(b) For any 1-form \( \alpha = F_1 \, dx + F_2 \, dx + F_3 \, dx \),
\[
 d\alpha = (dF_1) \wedge dx + (dF_2) \wedge dy + (dF_3) \wedge dz
\]
where, to expand the 2-form to its standard description, we distribute across the wedge product, and use the “repeating any \( dx_i \) means zero” and “swapping two positions changes the signs” rules. Applied to the 1-form
\[
 \alpha = (y \cos(xy) - 3y^2z) \, dx + (x \cos(xy) - 6xyz) \, dy + (-3xy^2) \, dz
\]
from part (a), this gives
\[ d\alpha \] 
\[ = \left( \begin{aligned} &d \left( y \cos(xy) - 3y^2z \right) \wedge dx + d \left( x \cos(xy) - 6xyz \right) \wedge dy + d \left( -3xy^2 \right) \wedge dz \\ &\end{aligned} \right) \]
\[ = \left( \begin{aligned} &d \left( -y^2 \sin(xy) \right) dx + \left( \cos(xy) - xy \sin(xy) - 6yz \right) dy + \left( -3y^2 \right) dz \wedge dx \\ &+ \left( \cos(xy) - xy \sin(xy) - 6yz \right) dx \wedge dy + \left( -x^2 \sin(xy) - 6xz \right) dy \wedge dy + \left( -6xy \right) dz \wedge dy \\ &+ \left( -3y^2 \right) dx \wedge dz + \left( -6xy \right) dy \wedge dz + 0 dz \wedge dz \\ &\end{aligned} \right) \]
\[ = \left( \begin{aligned} &\left( -y^2 \sin(xy) - 3y^2 \right) dx \wedge dx + \left( \cos(xy) - x^2 \sin(xy) - 6xz \right) dy \wedge dx + \left( -3y^2 \right) dz \wedge dx \\ &+ \left( \cos(xy) - xy \sin(xy) - 6yz \right) dx \wedge dy + \left( -x^2 \sin(xy) - 6xz \right) dy \wedge dy + \left( -6xy \right) dz \wedge dy \\ &+ \left( -3y^2 \right) dx \wedge dz + \left( -6xy \right) dy \wedge dz + 0 \\ &\end{aligned} \right) \]
\[ = \left( \begin{aligned} &0 + \left( -\cos(xy) + x^2 \sin(xy) + 6xz \right) dx \wedge dy + \left( 3y^2 \right) dx \wedge dz \\ &+ \left( \cos(xy) - xy \sin(xy) - 6yz \right) dx \wedge dy + 0 + \left( 6xy \right) dy \wedge dz \\ &+ \left( -3y^2 \right) dx \wedge dz + \left( -6xy \right) dy \wedge dz + 0 \\ &\end{aligned} \right) \]
\[ = \left( \begin{aligned} &\left( -\cos(xy) + x^2 \sin(xy) + 6xz \right) + \left( \cos(xy) - xy \sin(xy) - 6yz \right) \\ &+ \left( 3y^2 \right) + \left( -3y^2 \right) \\ &+ \left( 6xy \right) + \left( -6xy \right) \\ &\end{aligned} \right) dy \wedge dz 
\[ = \left( \begin{aligned} &0 \\ &0 dx \wedge dy + 0 dx \wedge dz + 0 dy \wedge dz \\ &\end{aligned} \right) \]
\[ = 0. \]

The steps in each of these equalities are:

1. The definition of \( d \) of a 1-form;
2. Working out \( dF_1, dF_2 \), and \( dF_3 \);
3. Distributing across the wedge product;
4. Using the rules “any two repeated terms means zero” (i.e., \( dx \wedge dx = 0 \), \( dy \wedge dy = 0 \), and \( dx \wedge dz = 0 \)), as well as “swapping any two changes the sign” (i.e., \( dy \wedge dx = -dx \wedge dy \), \( dz \wedge dx = -dx \wedge dz \), and \( dz \wedge dy = -dy \wedge dy \));
5. Collecting the terms in front of \( dx \wedge dy \), \( dx \wedge dz \), and \( dy \wedge dz \);
6. Simplifying each of the expressions;
7. ... that’s what the zero 2-form means.
(c) Starting with a vector field \( \mathbf{G} = (G_1, G_2, G_3) \), we convert it to a 2-form according to the rule above, to get \( \beta = G_1 \, dy \wedge dz - G_2 \, dx \wedge dz + G_3 \, dx \wedge dy \). Taking \( d \) of this gives

\[
d\beta \overset{(1)}{=} (dG_1) \wedge dy \wedge dz - (dG_2) \wedge dx \wedge dz + (dG_3) \, dx \wedge dy
\]

\[
\overset{(2)}{=} \left( G_{1,x} \, dx + G_{1,y} \, dy + G_{1,z} \, dz \right) \wedge dy \wedge dz + \left( -G_{2,x} \, dx - G_{2,y} \, dy - G_{2,z} \, dz \right) \wedge dx \wedge dz
\]

\[
\quad + \left( G_{3,x} \, dx + G_{3,y} \, dy + G_{3,z} \, dz \right) \wedge dx \wedge dy
\]

\[
\overset{(3)}{=} G_{1,x} \, dx \wedge dy \wedge dz + G_{1,y} \, dy \wedge dy \wedge dz + G_{1,z} \, dz \wedge dy \wedge dz
\]

\[
- G_{2,x} \, dx \wedge dx \wedge dz - G_{2,y} \, dy \wedge dx \wedge dz - G_{2,z} \, dz \wedge dx \wedge dz
\]

\[
+ G_{3,x} \, dx \wedge dx \wedge dy + G_{3,y} \, dy \wedge dx \wedge dy + G_{3,z} \, dz \wedge dx \wedge dy
\]

\[
\overset{(4)}{=} G_{1,x} \, dx \wedge dy \wedge dz + 0 + 0
\]

\[
0 + G_{2,y} \, dx \wedge dy \wedge dz
\]

\[
0 + 0 + G_{3,z} \, dx \wedge dy \wedge dz
\]

\[
\overset{(5)}{=} \left( G_{1,x} + G_{2,y} + G_{3,z} \right) \, dx \wedge dy \wedge dz.
\]

Here the steps were

(1) The definition of \( d \) of a 2-form;
(2) Working out \( dG_1 \), \( dG_2 \), and \( dG_3 \);
(3) Distributing across the wedge product;
(4) Using the rules “any two repeated terms means zero” (i.e., \( dx \wedge dx \wedge dy = 0 \), \( dx \wedge dy \wedge dx = 0 \), \( dy \wedge dz \wedge dy = 0 \), \ldots ), as well as “swapping any two changes the sign” (i.e., \( dy \wedge dx \wedge dz = -dx \wedge dy \wedge dz \), or \( dz \wedge dx \wedge dy = -dx \wedge dz \wedge dy \) = \( dx \wedge dy \wedge dz \));
(5) Collecting the terms in front of \( dx \wedge dy \wedge dz \).

Using our dictionary between 3-forms and functions, this is the function \( G_{1,x} + G_{2,y} + G_{3,z} \), also known as \( \text{Div}(\mathbf{G}) \).
(d) Now starting with the vector field \( \mathbf{F} = (F_1, F_2, F_3) \), we convert it to the one form 
\( \gamma = F_1 \, dx + F_2 \, dy + F_3 \, dz \), and then compute that

\[
d\gamma \overset{(1)}{=} (dF_1) \wedge dx + (dF_2) \wedge dy + (dF_3) \wedge dz
\]
\[
\overset{(2)}{=} \left( F_{1,x} \, dx + F_{1,y} \, dy + F_{1,z} \, dz \right) \wedge dx
\]
\[+ \left( F_{2,x} \, dx + F_{2,y} \, dy + F_{2,z} \, dz \right) \wedge dy
\]
\[+ \left( F_{3,x} \, dx + F_{3,y} \, dy + F_{3,z} \, dz \right) \wedge dz
\]
\[
\overset{(3)}{=} F_{1,x} \, dx \wedge dx + F_{1,y} \, dy \wedge dx + F_{1,z} \, dz \wedge dx
\]
\[+ F_{2,x} \, dx \wedge dy + F_{2,y} \, dy \wedge dy + F_{2,z} \, dz \wedge dy
\]
\[+ F_{3,x} \, dx \wedge dz + F_{3,y} \, dy \wedge dz + F_{3,z} \, dz \wedge dz
\]
\[
\overset{(4)}{=} 0 - F_{1,y} \, dx \wedge dy - F_{1,z} \, dx \wedge dz
\]
\[+ F_{2,x} \, dx \wedge dy + 0 - F_{2,z} \, dy \wedge dz
\]
\[+ F_{3,x} \, dx \wedge dz + F_{3,y} \, dy \wedge dz + 0
\]
\[
\overset{(5)}{=} \left( F_{3,y} - F_{2,z} \right) \, dy \wedge dz + \left( F_{3,x} - F_{1,z} \right) \, dx \wedge dz + \left( F_{2,x} - F_{1,y} \right) \, dx \wedge dy
\]
\[
\overset{(6)}{=} \left( F_{3,y} - F_{2,z} \right) \, dy \wedge dz - \left( F_{1,z} - F_{3,x} \right) \, dx \wedge dz + \left( F_{2,x} - F_{1,y} \right) \, dx \wedge dy.
\]

The equalities are:

1. The definition of \( d \) of a 1-form;
2. Working out \( dF_1, dF_2 \), and \( dF_3 \);
3. Distributing across the wedge product;
4. Using the rules “any two repeated terms means zero” and “swapping any two changes the sign”;
5. Collecting the terms in front of \( dx \wedge dy, dx \wedge dz, \) and \( dy \wedge dz \);
6. Organizing the signs to match the pattern for the (2-form) ↔ (vector field) dictionary.

Translating the 2-form \( d\gamma \) into a vector field using our dictionary, this is the vector field

\[
\left( F_{3,y} - F_{2,z}, F_{1,z} - F_{3,x}, F_{2,x} - F_{1,y} \right) = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = \text{Curl}(\mathbf{F}).
\]