1. Let \( f \) be a function \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \), and \( \mathbf{F} \) and \( \mathbf{G} \) vector fields on \( \mathbb{R}^3 \) (i.e, functions \( \mathbb{R}^3 \rightarrow \mathbb{R}^3 \)). State whether each of the following expressions is a function from \( \mathbb{R}^3 \) to \( \mathbb{R} \), a vector field, (i.e., a function from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \)), or meaningless.

(a) \( \text{grad}(\text{grad}(f)) \)  
(b) \( \text{Curl}(\text{grad}(f)) - \mathbf{F} \)  
(c) \( \text{Curl}(\text{Curl}(\mathbf{F})) - \mathbf{G} \)  
(d) \( \text{Curl}(\mathbf{F}) \cdot \mathbf{G} \)  
(e) \( \text{Div}(\text{Div}(\mathbf{F})) \)  
(f) \( \text{Div}(\text{Curl}(\text{grad}(f))) \)

2. Compute \( \text{Div} \) and \( \text{Curl} \) for the following vector fields:

(a) \( \mathbf{F}(x, y, z) = (x, y, z) \)  
(b) \( \mathbf{F}(x, y, z) = (yz, xz, xy) \)  
(c) \( \mathbf{F}(x, y, z) = (3x^2y, x^3 + y^3, z^4) \)  
(d) \( \mathbf{F}(x, y, z) = (e^x \cos(y) + z^2, e^x \sin(y) + xz, xy) \)

3. The vector field \( \vec{r}(x, y, z) = (x, y, z) \) is commonly used in physics. Let \( a, b, c \in \mathbb{R} \) be any real numbers, and set \( \vec{w} = (a, b, c) \).

(a) Compute \( \vec{r} \times \vec{w} \).

(b) Compute the curl of your answer from (a).

4. 

(a) Is there a vector field \( \mathbf{F} \) such that \( \text{Curl}(\mathbf{F}) = (xy^2, yz^2, zx^2) \)? Explain.

(b) Is there a vector field \( \mathbf{F} \) so that \( \text{Curl}(\mathbf{F}) = (2, 1, 3) \)? If so, find one.

5. A vector field \( \mathbf{F} \) is called \textit{incompressible} if \( \text{Div}(\mathbf{F}) = 0 \), and \textit{irrotational} if \( \text{Curl}(\mathbf{F}) = 0 \).

(a) Show that any vector field of the form \( \mathbf{F}(x, y, z) = (f(x), g(y), h(z)) \) is irrotational.

(b) Show that any vector field of the form \( \mathbf{F}(x, y, z) = (f(y, z), g(x, z), h(x, y)) \) is incompressible.

(c) Find constants \( a, b, \) and \( c \) so that the vector field \( \mathbf{F}(x, y, z) = (3x - y + az, bx - z, 4x + cy) \) is irrotational. For these values of \( a, b, \) and \( c \), find a function \( f \) with \( \nabla f = \mathbf{F} \).
The Laplacian of a function $f: \mathbb{R}^n \to \mathbb{R}$ is defined to be $\text{Div}(\text{grad}(f))$, and is denoted $\Delta f$. (Note: $\Delta$ may look a bit like the gradient, $\nabla$, but it isn’t – it’s the Laplacian.) Composing the gradient and the divergence, we compute that for a function $f: \mathbb{R}^n \to \mathbb{R}$,

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$ 

A function $f$ is called harmonic if $\Delta f = 0$.

(d) If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ which is both incompressible and irrotational, show that $\mathbf{F}$ is the gradient of a harmonic function $f$. 

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