1. Let $c$ be the top half of the unit circle, oriented from $(-1, 0)$ to $(1, 0)$, and $\mathbf{F}$ be the vector field
$$\mathbf{F}(x, y) = (\ln(x + 5) + y^2, 2xy - y^2).$$
Check that $\text{Curl}_2(\mathbf{F}) = 0$ and use the flexibility theorem for curves to compute the integral $\int_c \mathbf{F} \cdot ds$. Flexing the curve to the line joining $(-1, 0)$ and $(1, 0)$ is a possibility.

2. Let $\mathbf{F}$ be the vector field
$$\mathbf{F}(x, y, z) = (ye^{z^2} - xe^{xy}, ye^{xy} + \tan(z^2 + z + 1), x^2).$$
Check that $\text{Div}(\mathbf{F}) = 0$, and use the flexibility theorem for surfaces to compute $\iint_S \mathbf{F} \cdot dS$ where $S$ is the top half of the unit sphere oriented outwards. (“Flexing” it to the unit disk in the $xy$-plane seems like a good bet.)

3. Let $c_1$ be the top half of the unit circle, oriented counterclockwise, $c_2$ be the line segment joining $(1, 0)$ to $(-1, 0)$, oriented from $(1, 0)$ to $(-1, 0)$, and let $\mathbf{F}$ be the vector field
$$\mathbf{F}(x, y) = (x^2 - y, xy - \arcsin(y) + e^{y^3}).$$
(a) Use Green’s theorem to compute the difference between $\int_{c_1} \mathbf{F} \cdot ds$ and $\int_{c_2} \mathbf{F} \cdot ds$.
(b) Use (a) to compute $\int_{c_1} \mathbf{F} \cdot ds$ (by computing the easier $\int_{c_2} \mathbf{F} \cdot ds$, of course...).

4. Questions about differential forms.
(a) Let $f = \sin(xy) - 3xy^2z$. Compute the 1-form $df$.

It turns out that doing $d$ twice in a row always results in 0. The reason is a combination of “mixed partials commute”, and the rules for differentials (“swapping any two $dx_i$, $dx_j$ changes the sign”, and “any repeated $dx_i$ means that the form is zero”).

(b) Let $\alpha$ be your answer from (a). Check the claim above by computing the 2-form $d\alpha$. I.e., compute $d\alpha$ and check that it is zero, showing the details.
The $d$ operators fit together to give a diagram

$$
\left\{ \text{Functions } R^3 \to R \right\} \xrightarrow{d} \left\{ \text{1-forms on } R^3 \right\} \xrightarrow{d} \left\{ \text{2-forms on } R^3 \right\} \xrightarrow{d} \left\{ \text{3-forms on } R^3 \right\},
$$

where any two in a row is zero.

You have a friend who is does not like differential forms. Fortunately you have a great idea on how to help them. You think : “A 1-form is something that looks like $F_1 \, dx + F_2 \, dy + F_3 \, dz$, where $F_1$, $F_2$, and $F_3$ are functions on $R^3$. Another way to keep track of three different functions is a vector field $F = (F_1, F_2, F_3)$ with three different components. I’ll tell my friend to forget about 1-forms, and explain everything using vector fields.”

For instance, given a function $f$, you know that $df = f_x \, dx + f_y \, dy + f_z \, dz$. In terms of your (vector field) $\leftrightarrow$ (1-form) dictionary, this is the vector field $(f_x, f_y, f_z)$. You then tell your friend : “Don’t worry about the $d$ operator that takes functions to 1-forms. If you have a function $f$, $d$ of that is just the vector field $(f_x, f_y, f_z)$”.

Encouraged by your success with 1-forms, you decide to simplify 2-forms and 3-forms too. A 2-form is also given by three functions, the entries in front of $dx \wedge dy$, $dy \wedge dz$, and $dx \wedge dz$. You decide to match this up with a vector field by declaring that the vector field $G = (G_1, G_2, G_3)$ corresponds to the 2-form $G_1 \, dy \wedge dz - G_2 \, dx \wedge dz + G_3 \, dx \wedge dy$. [This choice of signs will make more sense after (c) and (d) below.] You also realize that 3-forms are just multiples of $dx \wedge dy \wedge dz$, and so can be described by a single function (e.g., the 3-form $H \, dx \wedge dy \wedge dz$ would correspond to the function $H$).

(c) Using your dictionary, tell your friend how the $d$ operator takes 2-forms to 3-forms. I.e., starting with $G = (G_1, G_2, G_3)$ convert $G$ to a 2-form by the rule above, apply $d$ to that to get a 3-form, and then convert the 3-form back to a function (showing the details of your calculations). What function (in terms of $G_1$, $G_2$, and $G_3$) do you get?

(d) Finally, your toughest challenge : explain to your friend how to take $d$ of a 1-form using your dictionary. I.e., start with a vector field $F = (F_1, F_2, F_3)$, convert it to a 1-form, take $d$ of that 1-form to get a 2-form, and then convert the 2-form back to a vector field $G$ (all using the rules above). Starting with $F$, what is the formula for the vector field $G$ that results?

This homework assignment is due on or before Thursday, December 7th, at 4pm. The homework should be handed in to the mailbox of either Mike Roth (507 Jeffery Hall), or Kexue Zhang (419 Jeffery Hall).