1. Draw pictures of the zero loci of the two equations \( f_1 = xz - x \) and \( f_2 = x^2 + y^2 - z^2 \) in \( \mathbb{A}^3 \). Find their intersection and decompose it into irreducible components. Find the prime ideals in \( k[x, y, z] \) associated to each component.

**Solution.** The zero locus of \( f_1 = xz - x = x(z - 1) \) consists of the plane \( x = 0 \) and the plane \( z = 1 \). The zero locus of \( f_2 = x^2 + y^2 - z^2 \) is the cone from Homework 2, Question 1(a). Their intersection looks like this:

![Diagram of zero loci](image)

The components are the circle, with equations \( z = 1, x^2 + y^2 = 1 \), the line \( x = 0, y = z \), and the line \( x = 0, y = -z \). The respective ideals are \((z - 1, x^2 + y^2 - 1), (x, y - z), \) and \((x, y + z)\).

2. Draw pictures of the various kinds of irreducible subvarieties in \( \mathbb{A}^3 \), analogous to the one we drew in class for \( \mathbb{A}^2 \). Include a parallel diagram of corresponding prime ideals.

**Solution.** The picture appears on the next page. In that diagram, the maximal ideals \( m_1, m_2, \) and \( m_3 \) correspond to points \( p_1, p_2, \) and \( p_3 \); the prime ideals \( Q_1 \) and \( Q_2 \) correspond to curves \( C_1 \) and \( C_2 \); prime ideals \( P_1, P_2, \) and \( P_3 \) correspond to surfaces \( S_1, S_2, \) and \( S_3 \); finally the prime ideal \((0)\) corresponds to all of \( \mathbb{A}^3 \). The inclusion among the ideals corresponds to the inclusions among the irreducible subvarieties, but in the opposite direction.

We can read a few more facts off from this diagram. In the picture the curve \( C_2 \) is the intersection of the two surfaces \( S_2 \) and \( S_3 \); the corresponding relation among the prime ideals is \( Q_2 = \sqrt{P_2 + P_3} \). However, \( C_1 \) is not the intersection of \( S_1 \) and \( S_2 \), that intersection has at least one other component. Therefore, the equations in \( P_1 \) and \( P_2 \) are not enough to generate \( Q_2 \) (not even up to radical), more equations are needed to get rid of the other component.
3. Let $X = \mathbb{A}^n$ and for $1 \leq s \leq n$ let $V_s$ be the open subset of $X$ which is the complement of the linear space $x_1 = x_2 = x_3 = \cdots = x_s = 0$. Compute (analogously to the computation for $\mathbb{A}^2$ and $s = 2$) the ring of functions $\mathcal{O}_X(V_s)$. (You can make your life easier in the case $s > 2$ by appealing to your answer for $s = 2$.)

**Solution.** First consider the case that $s = 2$. Then $V_2$ is covered by the two principal open sets $x_1 \neq 0$ and $x_2 \neq 0$ with respective coordinate rings $k[x_1, \ldots, x_n, \frac{1}{x_1}, \frac{1}{x_2}]$ and $k[x_1, \ldots, x_n, \frac{1}{x_2}]$. By construction, an element of $\mathcal{O}_{\mathbb{A}^n}(V_2)$ is a pair $g_1 \in \mathcal{O}_{\mathbb{A}^n}(U_{x_1})$ and $g_2 \in \mathcal{O}_{\mathbb{A}^n}(U_{x_2})$ which agree on the intersection. Write $g_1$ as a polynomial in $x_1$ and $x_2$ whose coefficients are in $k[x_3, x_4, \ldots, x_n]$, i.e. as

$$g_1 = \sum b_{ij}(x_3, \ldots, x_i)x_1^ix_2^j$$

with $b_{ij}(x_2, \ldots, x_n) \in k[x_3, \ldots, x_n], j \geq 0$ and $i \in \mathbb{Z}$. Similarly we can write

$$g_2 = \sum c_{ij}(x_3, \ldots, x_n)x_1^ix_2^j$$

with $c_{ij}(x_2, \ldots, x_n) \in k[x_3, \ldots, x_n], i \geq 0$ and $j \in \mathbb{Z}$.

In order for $g_1$ and $g_2$ to agree in $\mathcal{O}_{\mathbb{A}^n}(U_{x_1x_2}) = k[x_1, \ldots, x_n, \frac{1}{x_1}, \frac{1}{x_2}]$, the coefficients of each monomial $x_1^ix_2^j$ must agree. That is, we must have $b_{ij} = c_{ij}$ for all $i$ and $j$. Since $c_{ij} = 0$ when $i < 0$, we have $b_{ij} = 0$ for negative $i$ as well. Thus $b_{ij} \neq 0$ only for nonnegative $i$ and $j$. By the equality $c_{ij} = b_{ij}$ the same is true for $c_{ij}$. Thus both $g_1$ and $g_2$ are polynomials in $x_1, \ldots, x_n$ (and the same polynomial). Therefore $\mathcal{O}_{\mathbb{A}^n}(V_2) = k[x_1, \ldots, x_n]$.

To deal with the case $s > 2$, we could repeat this type of computation, or take a shortcut. We have $V_2 \subset V_3 \subset V_4 \subset \cdots \subset V_n$. Hence, when $s > 2$ we have an inclusion $V_2 \subset V_s$, and therefore a restriction map $\mathcal{O}_{\mathbb{A}^n}(V_s) \rightarrow \mathcal{O}_{\mathbb{A}^n}(V_2)$. We also have a restriction map $\mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n) \rightarrow \mathcal{O}_{\mathbb{A}^n}(V_s)$. By the first part of this problem, the composite map

$$k[x_1, \ldots, x_n] = \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n) \rightarrow \mathcal{O}_{\mathbb{A}^n}(V_s) \rightarrow \mathcal{O}_{\mathbb{A}^n}(V_2) = k[x_1, \ldots, x_n]$$

is an isomorphism. Since $\mathbb{A}^n$ is a domain, the restriction map $\mathcal{O}_{\mathbb{A}^n}(V_s) \rightarrow \mathcal{O}_{\mathbb{A}^n}(V_2)$ is an inclusion. By the above composition map, above, the map $\mathcal{O}_{\mathbb{A}^n}(V_s) \rightarrow \mathcal{O}_{\mathbb{A}^n}(V_2)$ is a surjection as well. Thus the map $\mathcal{O}_{\mathbb{A}^n}(V_s) \rightarrow \mathcal{O}_{\mathbb{A}^n}(V_2)$ is an isomorphism, and so $\mathcal{O}_{\mathbb{A}^n}(V_s) = \mathcal{O}_{\mathbb{A}^n}(V_2) = k[x_1, \ldots, x_n]$. \qed

4. Let $X$ be the affine variety described by the equation $xy - z^2 = 0$ in $\mathbb{A}^3$, and let $U \subset X$ be the complement of $(0,0,0) \in X$. In this problem we will compute $\mathcal{O}_X(U)$ and confirm that it is equal to $R[X]$.

The variety $X$ is covered by the principal open sets $U_x$ and $U_y$, with coordinate rings $k[x, y, z, 1/x]/(xy - z^2) \cong k[x, 1/x, z]$ and $k[x, y, z, 1/y]/(xy - z^2) \cong k[y, 1/y, z]$ respectively. Any function $g_1 \in R(U_x)$ can be written as a finite sum $g_1 = \sum a_{ij}x^iz^j$ and any function $g_2 \in R(U_y)$ can be written as a finite sum $g_2 = \sum b_{kel}y^kz^l$. 

3
(a) What range of indices are valid in the expressions for \( g_1 \) and \( g_2 \) above?

We want to look at pairs \((g_1, g_2)\) which agree on \( U_x \cap U_y \). The expressions for \( g_1 \) and \( g_2 \) above are with respect to different variables. To compare them we need to write them in terms of the same variables.

(b) Use the relation \( y = \frac{z^2}{x} \) (valid on \( U_x \), and hence also on \( U_x \cap U_y \)) to write \( g_2 \) in terms of the variables \( x \) and \( z \).

(c) In order for \( g_1 \) to be equal to \( g_2 \), what must be the relation between the \( a_{ij} \) and the \( b_{k\ell} \)?

(d) Considering the restrictions on the indices from part (a), your formula from (c) will imply additional restrictions on \( i \) and \( j \). What are they?

(e) For each \( i \) and \( j \) satisfying the conditions above, show that there is a monomial \( x^{p}y^{q}z^{r} \) which is equal to \( x^{i}z^{j} \) on \( U_x \).

(f) Explain why this means that the restriction homomorphism \( R[X] \to \mathcal{O}_X(U) \) is surjective.

**Solution.**

(a) The ranges are \( i \in \mathbb{Z} \) and \( j \geq 0 \) for \( g_1 \) and \( k \in \mathbb{Z} \) and \( \ell \geq 0 \) for \( g_2 \).

(b) \[ g_2 = \sum b_{k\ell} y^{k} z^{\ell} = \sum b_{k\ell} \left( \frac{z^2}{x} \right)^{k} z^{\ell} = \sum b_{k\ell} x^{-k} z^{\ell+2k}. \]

(c) In order for \( g_1 \) and \( g_2 \) to agree, the coefficients of each monomial must match up. Thus we must have \( b_{k\ell} = a_{-k,\ell+2k} \) for all \( k \) and \( \ell \), or reversing the formula, that \( a_{ij} = b_{-i,j+2i} \).

(d) Since \( b_{k\ell} = 0 \) when \( \ell < 0 \), we have \( a_{ij} = 0 \) for \( j + 2i < 0 \). Thus nonzero \( a_{ij} \) occur only when \( j + 2i \geq 0 \). Here is a picture of the pairs \((i, j)\) satisfying this condition (as well as \( j \geq 0, i \in \mathbb{Z} \)):
(e) Consider a monomial \( x^i y^j \) with \( i \in \mathbb{Z}, j \geq 0 \), and \( j + 2i \geq 0 \). If \( j \) is even set \( q = \frac{j}{2} \) and \( p = i + \frac{j}{2} \). Then \( q \geq 0 \) since \( j \geq 0 \), and \( p \geq 0 \) since \( 2p = 2i + j \geq 0 \). The monomial \( x^p y^q \) therefore belongs to \( \mathcal{O}_X(X) \). Its restriction to \( U_x \) is equal to

\[
x^p \left( \frac{z^2}{x} \right)^q = x^{p-q} z^{2q} = x^{(i+\frac{j}{2})-\frac{j}{2}} z^{2(\frac{j}{2})} = x^i y^j.
\]

On the other hand, if \( j \) is odd, set \( q = \frac{j-1}{2} \), \( p = i + \frac{j-1}{2} \). Since \( j \geq 0 \) and is odd, \( j + 1 \) and therefore \( q \geq 0 \). Since \( 2i + j \) is \( \geq 0 \) and odd (\( j \) is odd, and \( 2i \) even), \( 2i + j \geq 1 \) and therefore \( 2i + j \geq 1 \) and so \( 2p = 2i + (j-1) \geq 0 \). The monomial \( x^p y^q z \) therefore belongs to \( \mathcal{O}_X(X) \). Its restriction to \( U_x \) is

\[
x^p \left( \frac{z^2}{x} \right)^q z = x^{p-q} z^{2q+1} = x^{(i+\frac{j-1}{2})-\frac{j-1}{2}} z^{2(\frac{j-1}{2})+1} = x^i y^j.
\]

(f) Given any \( g_1 \) and \( g_2 \) which agree on \( U_{xy} \) as above, part (c) shows us that \( g_1 = \sum a_{ij} x^i y^j \) with \( i \in \mathbb{Z}, j \geq 0 \), and \( j + 2i \geq 0 \). By part (e) any such monomial \( x^i y^j \) is the restriction of a monomial of the form \( x^p y^q z^r \), with \( p, q, r \geq 0 \), is the restriction of something in \( R[X] = \mathcal{O}_X(X) \). Thus the restriction map \( R[X] = \mathcal{O}_X(X) \to \mathcal{O}_X(U) \) is surjective.

5. Given a ring \( A \) and an element \( f \in A \) we have been looking at the ring \( A[1/f] \) obtained by adjoining the additional element \( 1/f \) to \( A \) (and of course using ring operations to get more elements). More precisely the ring \( A[1/f] \) is the ring \( A[y]/((1-yf)) \). There is a natural ring homomorphism \( A \to A[1/f] \), and we have seen in class that this is not always injective. For instance, if \( h \in A \) is an element so that \( h\cdot f^n = 0 \) for some \( n \geq 1 \), then in \( A[1/f] \) we compute that \( h = (h \cdot f^n) \cdot \frac{1}{f^n} = 0 \cdot \frac{1}{f^n} = 0 \).

The purpose of this question is to prove the converse direction: An element \( h \in A \) is in the kernel of the map \( A \to A[1/f] \) only if there is an \( n \geq 1 \) such that \( h \cdot f^n = 0 \) in \( A \).

Suppose that \( h \) is such an element. This means that the image of \( h \) under the inclusion \( A \hookrightarrow A[y] \) must be in the ideal \( (yf-1) \) in \( A[y] \). Therefore there is a polynomial \( g \in A[y] \) such that \( h = g(yf-1) \). Since \( g \in A[y] \) we can write \( g \) as \( g = g_0 + g_1 y + g_2 y^2 + \cdots + g_n y^n \) with each \( g_j \in A \).

(a) Expand \( g \cdot (yf-1) \) as a polynomial in \( y \).

(b) As a polynomial in \( y \), \( h \) has degree 0. Since we have \( h = g \cdot (yf-1) \), the coefficients of powers of \( y \) on both sides of the equality must be the same. Comparing coefficients, write down all the relations you obtain.

(c) Show that \( h \cdot f^{n+1} = 0 \).
Solution.

(a) \[ g(yf - 1) = (g_0 + g_1y + g_2y^2 + \cdots + g_ny^n)(yf - 1) \]
\[ = -g_0 + (fg_0 - g_1)y + (fg_1 - g_2)y^2 + \cdots + (fg_{n-1} - g_n)y^n + (fg_n)y^{n+1}. \]

(b) Comparing powers of \( y \) we have:

\[
\begin{align*}
    h &= -g_0, \\
    0 &= fg_0 - g_1, \\
    0 &= fg_1 - g_2, \\
    & \vdots \\
    0 &= fg_{n-1} - g_n, \\
    0 &= fg_n.
\end{align*}
\]

(c) From the second through last equations we have \( g_1 = fg_0, \ g_2 = fg_1 = f^2g_0, \ g_3 = fg_2 = f^3g_0, \ldots, \ g_n = fg_{n-1} = f^n g_0, \) and finally \( 0 = fg_n = f^{n+1}g_0. \) From the first equation we have \( h = -g_0, \) and therefore \( f^{n+1} \cdot h = -f^n \cdot g_0 = 0. \)