1. Suppose that $X \subset \mathbb{R}^n$ is a shape.

(a) If $f_1$ and $f_2$ are functions on $\mathbb{R}^n$, show that $f_1 = f_2$ on $X$ (i.e., when restricted to $X$) if and only if $f_1 - f_2$ is zero on $X$.

(b) If $g$ is a function on $\mathbb{R}^n$ which is zero when restricted to $X$, and $h$ any function on $\mathbb{R}^n$, show that $hg$ is zero when restricted to $X$.

(c) Now let $X$ be the circle $\{(x, y) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Take the following functions on $\mathbb{R}^2$ and organize them into groups according to their equality when restricted to $X$:

$$(1) \ 1; \quad (2) \ y; \quad (3) \ x^2 + y^2; \quad (4) \ x^2 - y^2;$$

$$(5) \ 2x^2 + 1; \quad (6) \ 2x^2 - 1; \quad (7) \ x^4 - y^4; \quad (8) \ y^3 + x^2y.$$  

(I.e. group together the functions which are equal when restricted to $X$.)

(d) [Math 813 only] Let $X$ be the unit circle as in part (c). Let $f(x, y)$ be any polynomial in $x$ and $y$. Prove that there is a polynomial of the form $g(u, v, w) = g_0(u) + g_1(u)v$ such that the restriction of $f$ to $X$ is equal to the restriction of $g$ to $X$.

2. Let $X$ be the unit circle $\{(x, y) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$ and $Y$ the unit sphere $\{(u, v, w) \mid u^2 + v^2 + w^2 = 1\} \subset \mathbb{R}^3$. Define a map $\varphi : X \rightarrow Y$ by the rule $\varphi(x, y) = (xy, y^2, x)$.

(a) Show that $\varphi$ is well-defined. That is, show that if $(x, y) \in X$ then $\varphi(x, y) \in Y$.

(b) Compute $\varphi^*(u)$, $\varphi^*(v)$, and $\varphi^*(w)$.

(c) Compute $\varphi^*(3u^2 - 2uv + 5)$.

(d) Let $f$ be the function $5xy^3 + 7x^2 - 9y^2$ restricted to $X$. Find a polynomial $g(u, v, w)$ on $\mathbb{R}^3$ so that $f = \varphi^*(g)$.

3. Let $X = \mathbb{R}$ and $Y = \mathbb{R}^2$. The ring of polynomial functions on $X$ is $\mathbb{R}[x]$. The ring of polynomial functions on $Y$ is $\mathbb{R}[x, y]$.

(a) The ring $\mathbb{R}[x]$ is a subring of $\mathbb{R}[x, y]$, i.e., the inclusion map $\psi_1 : \mathbb{R}[x] \rightarrow \mathbb{R}[x, y]$ is a ring homomorphism. Find a map $\varphi_1 : Y \rightarrow X$ such that pullback by $\varphi_1$ induces $\psi_1$. (I.e., “$\varphi_1^* = \psi_1$”.)
(b) The map $\psi_2 : \mathbb{R}[x, y] \to \mathbb{R}[x]$ given by “setting $y = 0$” (i.e., $\psi_2(f(x, y)) = f(x, 0)$) is also a ring homomorphism. Find a map $\varphi_2 : X \to Y$ so that $\varphi_2^* = \psi_2$.

(c) How would you describe these maps geometrically? (I.e., in a picture or in words, what do they do?)

MINOR SUGGESTION: The fact that there is more than one $x$ may make things more confusing. Relabelling one set of variables and describing the ring homomorphisms in the new variables may make things a bit clearer.