Math 413/813 Homework Assignment 2

DUE DATE: JAN. 22, 2019

1. Draw sketches of the following varieties in $\mathbb{A}^3$ (with coordinates x, y, and z).

(a) $z^2 - x^2 - y^2 = 0$

(b) $y - x^2 = 0$.

(c) $(y - x^2)(z - 1) = 0$

(d) $x^2 + y^2 - 1 = 0$.

(e) $x^2 + y^2 - 1 = 0,\ z^2 - 1 = 0$.

(f) $x^2 + y^2 - 1 = 0,\ z^2 - x^2 - y^2 = 0$.

(Of course, you only have to draw the real points, i.e, solutions in $\mathbb{R}^3$.)

2. (COMPUTING IN QUOTIENT RINGS)

(a) Show that $\frac{k[x,y,z]}{\langle x^2-y, x^3-z\rangle} \cong k[x]$.

Recall that a ring A is called a *domain* if whenever a_1, $a_2 \in A$ are not zero, then $a_1 \cdot a_2 \neq 0$.

(b) Show that $A = \frac{k[x,y,z]}{\langle (y-x^2)(z-1)\rangle}$ is not a domain.

(c) Is $B = \frac{k[x,y,z]}{\langle x^2+y^2-1, z^2-x^2-y^2)\rangle}$ a domain?

NOTES: (1) To show that a ring is *not* a domain, you need to find two elements f_1 and f_2 of A such that $f_1 \neq 0$, $f_2 \neq 0$, but $f_1 f_2 = 0$. Since our rings are rings of functions on algebraic varieties, one way to show that a function is not zero is to evaluate it at a point of the corresponding variety. (2) You have already drawn pictures of the geometric shapes corresponding to the rings in 2(b,c).

3. (MORPHISMS)

(a) Let $\varphi\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^3$ be given by $\varphi(t) = (t, t^2, t^3)$. Show that the image of φ is contained in the variety defined by the equations $y - x^2 = 0,\ z - x^3 = 0$.

(b) Describe the ring homorphism from $k[x,y,z]/\langle y - x^2, z - x^3\rangle$ to $k[t]$ given by φ^*. (In particular, where do $\overline{x}$, $\overline{y}$, and $\overline{z}$ get sent?) Is φ^* surjective? Injective?

(c) Let X be $\{(u,v,w) \mid u^2+v^2+w^2 = 1\} \subset \mathbb{A}^3$, and Y the affine variety $\{(x,y,z,w) \mid xy - zw = 0\} \subset \mathbb{A}^4$. Does $\varphi = (1+u, 1-u, v+iw, v-iw)$ induce a map from X to Y? (Here i is the square root of -1.) If so analyze φ^* as in part (b).