1. Let $X$ and $Y$ be two affine varieties, with rings of functions $R[X]$ and $R[Y]$. In this problem we will use the theorem from the Jan. 23 and 25 classes to prove that $X$ and $Y$ are isomorphic varieties if and only if $R[X]$ and $R[Y]$ are isomorphic rings.

(a) Explain why $(1_X)^* = 1_{R[X]}$.

(b) Suppose that $\varphi: X \to X$ is a morphism of affine varieties and that $\varphi^* = 1_{R[X]}$. Explain why must have $\varphi = 1_X$.

(c) Suppose that $X$ and $Y$ are isomorphic affine varieties. Writing out the definition of “isomorphic varieties” and applying the functor to rings, explain why $R[X]$ and $R[Y]$ are isomorphic rings.

(d) Now suppose that $R[X]$ and $R[Y]$ are isomorphic rings. Write out the definition of “isomorphic rings” and use part (c) of the theorem as well as (b) above to show that $X$ and $Y$ are isomorphic varieties.

2. In this question we will see an example of a morphism of affine varieties which is a bijection on points, but which is not an isomorphism. (In other words, in the category of affine varieties, isomorphism implies more than just bijection.) Let $X = \mathbb{A}^1$ with ring of functions $k[t]$, and let $Y$ be the subset of $\mathbb{A}^2$ given by the equation $y^2 = x^3$.

(a) Let $\varphi: X \to \mathbb{A}^2$ be the map given by $\varphi(t) = (t^2, t^3)$. Show the image of $\varphi$ lies in $Y$, so that $\varphi$ defines a morphism $\varphi: X \to Y$.

(b) Show that $\varphi$ is surjective. (i.e., given $(x, y) \in Y$, show that there is a $t$ such that $\varphi(t) = (x, y)$.)

(c) Show that $\varphi$ is injective.

(d) Draw a sketch of $Y$ ($\mathbb{R}^2$ points only). One suggestion: from part (b) you know that $Y$ is the image of $\varphi$, so you can use the parameterization given by $\varphi$ to see what $Y$ looks like.

(e) Compute the image of the ring homomorphism $\varphi^*: R[Y] \to R[X]$ (and recall that $R[X] = k[t]$). Is $\varphi^*$ surjective?

(f) Explain why $\varphi$ is not an isomorphism of affine varieties.
3. Consider the following four affine varieties, all contained in \( \mathbb{A}^3 \).

\[
X = \left\{ (x_1, x_2, x_3) \mid x_1^2 + x_2^2 - 1 = 0 \right\} \subset \mathbb{A}^3
\]

\[
Y = \left\{ (y_1, y_2, y_3) \mid y_1^2 + y_2^2 - y_3^2 = 0 \right\} \subset \mathbb{A}^3
\]

\[
Z = \left\{ (z_1, z_2, z_3) \mid z_1^2 + z_2^2 + z_3^2 - 625 = 0 \right\} \subset \mathbb{A}^3
\]

\[
W = \left\{ (w_1, w_2, w_3) \mid w_1^2 + w_2^2 - w_3 = 0 \right\} \subset \mathbb{A}^3
\]

Define a map \( \varphi_1 : X \rightarrow \mathbb{A}^3 \) by \( \varphi_1(x_1, x_2, x_3) = (x_1x_3, x_2x_3, x_3) \).

(a) Draw sketches of \( X \), \( Y \), \( Z \), and \( W \).

(b) Is the image of \( \varphi_1 \) contained in \( Y \), \( Z \), or \( W \)? (Justify your answer.)

Define a map \( \varphi_2 : X \rightarrow \mathbb{A}^3 \) by \( \varphi_2(x_1, x_2, x_3) = (-9x_1 + 12x_2, 12x_1 - 16x_2, 20x_1 + 15x_2) \).

(c) Is the image of \( \varphi_2 \) contained in \( Y \), \( Z \), or \( W \)? (Justify your answer.)

Define a map \( \varphi_3 : Y \rightarrow \mathbb{A}^3 \) by \( \varphi_3(y_1, y_2, y_3) = (y_1, y_2, y_3^2) \).

(d) Is the image of \( \varphi_3 \) contained in \( X \), \( Z \), or \( W \)? (Justify your answer.)

One of the maps (b)–(d) has image in \( W \).

(e) What is the pullback of \( 3\overline{w}_1 - \overline{w}_2^2 + \overline{w}_3 \in R[W] \) under this map?

Now we will try and go the other way, from a map of rings to a map of varieties. Define a ring homomorphism

\[
R[X] = \frac{k[x_1, x_2, x_3]}{(x_1^2 + x_2^2 - 1)} \leftarrow \frac{k[w_1, w_2, w_3]}{(w_1^2 + w_2^2 - w_3)} = R[W] : \psi
\]

by the rule \( \psi(\overline{w}_1) = 2\overline{x}_1, \ \psi(\overline{w}_2) = 2\overline{x}_2, \ \psi(\overline{w}_3) = 4 \).

(f) Check that this ring homomorphism is well-defined by showing that \( \psi(\overline{w}_1^2 + \overline{w}_2^2 - \overline{w}_3) = 0 \).

(g) What geometric map \( \varphi : X \rightarrow W \) does the ring homomorphism \( \psi \) correspond to? (Write your formula for \( \varphi \) in the form \( \varphi(x_1, x_2, x_3) = (\text{formulas in } x_1, x_2, x_3) \subset \mathbb{A}^3 \) as in (b)–(d) above.)