1. A topological space \( X \) is called \textit{quasi-compact} if whenever \( \{U_i\}_{i \in S} \) are a family of open subsets such that \( \bigcup_{i \in S} U_i = X \) then there are a finite number of \( U_i \)'s which actually cover \( X \). (The term \textit{compact} is reserved for Hausdorff topological spaces with this finite subcover property. A topological space with the finite subcover property alone is called quasi-compact.) In this question we will show that affine varieties are quasi-compact.

We start by establishing a result we have used several times in class:

(a) Let \( X \) be an affine variety and \( f_i, i \in S \) a collection of elements of \( R[X] \). Show that the \( U_{f_i} (i \in S) \) cover \( X \) if and only if \( Z(\{f_i\}_{i \in S}) = \emptyset \). (The symbol \( Z(\{f_i\}_{i \in S}) \) means the subvariety of \( X \) where all the \( f_i \) are zero.)

Let \( A \) be a ring, \( f_i, i \in S \) elements of \( A \). The ideal generated by all the \( f_i \) is the set of elements of the form

\[
\left\{ \sum_{i \in S'} g_i f_i \middle| S' \subseteq S \text{ is a finite set, } g_i \in A \right\},
\]

i.e., the ideal consists of all finite linear combinations of the \( f_i \). (If \( S \) is finite, which has usually been the case for us, then a finite linear combination is the same as a linear combination, and so we haven’t noticed this distinction before.)

(b) Suppose that the ideal generated by the \( f_i, i \in S \) is all of \( A \). Show that there is a finite subset \( S' \subseteq S \) such that the ideal generated by the \( f_i, i \in S' \) is also \( A \). (Hint: An ideal is all of \( A \) if and only if 1 is in the ideal.)

(c) Show that if \( U_{f_i}, i \in S \) is a family of principal open subsets which cover an affine variety \( X \), then there is a finite number which cover \( X \). (Suggestion: Use parts (a), (b), and the Nullstellensatz.)

(d) Given an arbitrary cover of \( X \) by open sets \( \{U_j\}, j \in S \), use the fact that the principal open subsets are a basis for the Zariski topology and part (c) to show that a finite number of the \( U_j \) are sufficient to cover \( X \).

2. In class we have seen that if \( X \) is an affine variety and \( U_f \) a principal open subset, then \( U_f \) is an affine variety. Perhaps every open subset is an affine variety? The purpose of this question is to show that the answer to this is no. Let \( U = \mathbb{A}^2 \setminus \{(0,0)\} \). Recall that we have computed that \( \mathcal{O}_{\mathbb{A}^2}(U) = k[x, y] = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \).
(a) Let $\varphi : U \hookrightarrow \mathbb{A}^2$ be the inclusion map. Compute the ring homomorphism $\varphi^*$.

(b) Maps between affine varieties are completely determined by the pullback maps. If $U$ were an affine variety, explain why $\varphi$ would have to be an isomorphism.

(c) Show that $U$ is not an affine variety. (HINT : Is $\varphi$ an isomorphism?)

3. Although we are talking about $\mathbb{P}^n$ over algebraically closed fields, and usually over $\mathbb{C}$, we can consider $\mathbb{P}^n$ over any field. If we consider $\mathbb{P}^n$ over a finite field, then $\mathbb{P}^n$ only has finitely many points with coordinates in the field. In this problem we will count the number of points in two different ways. Let $p$ be a prime number.

(a) How many points does $\mathbb{A}^m$ have over $\mathbb{F}_p$?

(b) How many elements $\lambda \in \mathbb{F}_p$, $\lambda \neq 0$ are there?

(c) Considering $\mathbb{P}^n$ as $\mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\}$ modulo the relation of scaling by elements of $\mathbb{F}_p^*$, how many points does $\mathbb{P}^n$ have over $\mathbb{F}_p$?

(d) We have seen that the complement of a standard $\mathbb{A}^n$ coordinate chart in $\mathbb{P}^n$ is a $\mathbb{P}^{n-1}$. Continuing in this way we get a decomposition of $\mathbb{P}^n$ into disjoint subsets:

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{A}^{n-2} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0.$$  

Use this decomposition and part (a) to give a second formula for the number of points of $\mathbb{P}^n$ over $\mathbb{F}_p$.

(e) Check that your answers in (c) and (d) are the same.

(f) As a specific example, let $p = 2$. How many points does $\mathbb{P}^2$ have over $\mathbb{F}_2$? How many lines are there in $\mathbb{P}^2$ over $\mathbb{F}_2$? How many points are on each line?

REMARK. We could also have considered the case that the field is $\mathbb{F}_q$, with $q = p^r$ a prime power. The formulas, with $q$ taking the place of $p$, are the same.