1. In class we figured out the genus of a degree $d$ plane curve by taking the union of a degree $(d-1)$ curve and a line, and smoothing it. When seeing a new type of argument, it is good to check for consistency: If the argument is applied (correctly) in a similar way, it should also lead to correct conclusions. For instance, instead of smoothing a degree $(d-1)$ curve and a degree 1 curve, why not take the union of $d$ lines and smooth them?

(a) Suppose that $C_1, \ldots, C_r$ are curves in $\mathbb{P}^2$ of degrees $d_1, \ldots, d_r$. Show that their union is a curve of degree $d_1 + d_2 + \cdots + d_r$. (Suggestion: What is the definition of a “curve of degree $d$“?)

(b) Now let $C$ be the union of three distinct lines. By part (a) $C$ is a (singular) curve of degree 3. Draw a the real picture of an intersection of three lines. How many nodes does $C$ have? Draw the “balloon picture” of the nodal curve $C$, and then explain which genus Riemann surface is obtained when the curve is smoothed. Does this agree with our formula?

(c) Do the same thing for the union of four distinct lines in $\mathbb{P}^2$. You should suppose that the lines are general enough so that all the singularities are nodes. (For instance, while any pair of lines must intersect, three lines should never all meet in a single point.)

2. Find the unique singular point of the curve $6Y^2Z^2 = 6X^2Z^2 - 8X^3Z + 4Y^3Z + 3X^4$ in $\mathbb{P}^2$. Look at the equation in an affine chart of the singular point, and show that analytically it is a node. Draw a “balloon picture” of the topological shape of this curve. (Hints: The curve has degree 4, so you know what it looks like when it is smoothed. The curve is also irreducible, so only has one piece.)

3. Suppose that $C \subset \mathbb{P}^2$ is a curve, $q \in \mathbb{P}^2$ a point not on $C$, and $\ell$ a line not containing $q$. In class we saw how to use this setup to define a map $\varphi : C \rightarrow \ell$. (The procedure was: for any $p \in C$, let $\overrightarrow{pq}$ be the line containing $p$ and $q$, and define $\varphi(p)$ to be the intersection of $C$ and $\ell$.) In this question we will check that such a map is really a map of affine varieties.

We can make a useful simplification: We don’t really need to think about $C$ at all. Let $V = \mathbb{P}^2 \setminus \{q\}$. Then the procedure above really defines a map $\psi : V \rightarrow \ell$. The map $\varphi$ is the composite of $\psi$ with the inclusion $C \hookrightarrow V$. Since inclusion is an algebraic map, and compositions of algebraic maps are algebraic maps, all we really need to do is to verify that $\psi$ is an algebraic map.

Let $q = [0 : 0 : 1]$ and $\ell$ be the line $Z = 0$. 

(a) Let \( p = [\alpha : \beta : \gamma] \) be a point of \( V \). Write down the equation of the unique line in \( \mathbb{P}^2 \) which contains \( p \) and \( q \).

(b) Compute the intersection of the line above with \( \ell \) (i.e., calculate \( \psi(p) \)).

From your answer in (b), it will be clear that projection from \( q \) looks like an algebraic map. However, let’s practice computing in coordinates by examining this map in coordinate charts. The open set \( V \) is covered by the standard coordinate charts \( U_0 \) and \( U_1 \).

(c) Explain what the line \( \ell \) looks like in the coordinate system of \( U_0 \), and then write down the formula for the map \( U_0 \rightarrow (\ell \cap U_0) \) given by restricting \( \psi \) to \( U_0 \). (I.e., if \( p = (y_0, z_0) \) is a point of \( U_0 \), what point on \( (U_0 \cap \ell) \) is \( \psi(p) \)?)

(d) Do the same thing for \( U_1 \).