1. Here is an extremely simple example of a map between Riemann surfaces (aka “algebraic curves”). Fix an integer \( n \geq 1 \) and define a map \( \varphi: \mathbb{P}^1 \to \mathbb{P}^1 \) by the formula 
\[
[X:Y] \mapsto [X^n:Y^n].
\]

(a) Check that \( \varphi \) is well-defined, that is (1) \( \varphi \) doesn’t depend on the choice of representative we use for \([X:Y]\), and (2) no point of \( \mathbb{P}^1 \) is sent to \([0:0]\) by these instructions.

In order to see that this is a map of Riemann surfaces, let us look in coordinate charts.

(b) Check that \( \varphi^{-1}(U_0) = U_0 \) and that \( \varphi^{-1}(U_1) = U_1 \), i.e., that \( \varphi \) maps the standard coordinate charts to the standard coordinate charts.

(c) In each of \( U_0 \) and \( U_1 \) write out (in the coordinates of each chart) what \( \varphi \) is doing. Is \( \varphi \) an algebraic map?

(d) Find all the ramification points of \( \varphi \) and their ramification degrees.

The purpose of the next two questions is to think about the basic details of ramified covers, and to practice using the Riemann-Hurwitz formula.

2. Suppose that \( \pi: X \to Y \) is a map of degree \( d \) between Riemann surfaces.

(a) For any point \( p \) of \( X \), explain why the ramification index \( k_p \) cannot be bigger than \( d \), i.e., that \( k_p \leq d \).

(b) If \( d = 2 \) (a double cover) explain why a point \( p \) of \( X \) is either not a ramification point, or has ramification index exactly 2.

(c) Again in the case that \( d = 2 \) explain why the number of branch points (on \( Y \)) is the same as the number of ramification points (on \( X \)).

Parts (b) and (c) show that for degree 2 covers, the topological data (the number and type of ramification points) of a map \( \pi: X \to Y \) is given just by knowing the number of ramification or branch points. Use this to answer the following two questions.

(d) Suppose that \( \pi: X \to Y \) is a degree 2 map. Show that the number of ramification points is even.
(e) Suppose that $\pi : X \rightarrow \mathbb{P}^1$ is a double cover, with $2t$ branch points. What is the genus of $X$?

3. Use the Riemann-Hurwitz formula to find the genus of $X$, the genus of $Y$, or the number of ramification points, as required.

(a) $\pi : X \rightarrow \mathbb{P}^1$ is a degree 3 cover, with two ramification points, both with ramification index $k_p = 3$. Find the genus of $X$.

(b) $\pi : X \rightarrow \mathbb{P}^1$ is a degree 3 cover, with three ramification points, all with ramification index $k_p = 3$. Find the genus of $X$.

(c) $\pi : X \rightarrow Y$ is a map of degree $d$, $X$ has genus 1, and there are no ramification points. Find the genus of $Y$.

(d) $X$ is of genus $g$, $Y$ is of genus 1, the map $\pi : X \rightarrow Y$ is of degree $d$, and all ramification points $p$ in $X$ are of index 2. Find the number of ramification points (the answer turns out, in this case, not to depend on the degree $d$).

Can you think of a map $X \rightarrow \mathbb{P}^1$ satisfying the description in part (a)?