Math 413/813 Homework Assignment 11

DUE DATE: APR. 2, 2019

These questions will use the "Riemann-Hurwitz" formula, which will be proved in class on Thursday, March 28th. As well, in question 3 you will complete the proof of the 'global' picture of a map between Riemann surfaces, the statement of which which will also appear in Thursday's class.

1. Here is an extremely simple example of a map between Riemann surfaces (aka "algebraic curves"). Fix an integer $n \geqslant 1$ and define a map $\varphi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ by the formula $[X\colon Y] \longrightarrow [X^n\colon Y^n]$.

(*a*) Check that φ is well-defined, that is (1) φ doesn't depend on the choice of representative we use for $[X\colon Y]$, and (2) no point of $\mathbb{P}^1$ is sent to $[0\colon 0]$ by these instructions.

In order to see that this is a map of Riemann surfaces, let us look in coordinate charts.

(*b*) Check that $\varphi^{-1}(U_0) = U_0$ and that $\varphi^{-1}(U_1) = U_1$, i.e, that φ maps the standard coordinate charts to the standard coordinate charts.

(*c*) In each of U_0 and U_1 write out (in the coordinates of each chart) what φ is doing. Is φ an algebraic map?

(*d*) Find all the ramification points of φ and their ramification degrees.

2. Use the Riemann-Hurwitz formula to find the genus of X, the genus of Y, or the number of ramification points, as required.

(*a*) $\pi : X \longrightarrow \mathbb{P}^1$ is a degree 3 cover, with two ramification points, both with ramification index $k_p = 3$. Find the genus of X.

(*b*) $\pi : X \longrightarrow \mathbb{P}^1$ is a degree 3 cover, with three ramification points, all with ramification index $k_p = 3$. Find the genus of X.

(*c*) $\pi : X \longrightarrow Y$ is a map of degree d, X has genus 1, and there are no ramification points. Find the genus of Y.

(*d*) X is of genus g, Y is of genus 1, the map $\pi : X \longrightarrow Y$ is of degree d, and all ramification points p in X are of index 2. Find the number of ramification points (the answer turns out, in this case, not to depend on the degree d).

Can you think of a map $X \longrightarrow \mathbb{P}^1$ satisfying the description in part (a)?

3. In this question we will complete the proof of the theorem describing the "global" picture of a non-constant map $\varphi\colon X \longrightarrow Y$ between Riemann surfaces. The key missing step of the theorem was this : to show that there exists a positive integer d, such that for any $q \in Y$, $\sum_{p\in\varphi^{-1}(q)} k_p = d$. Here the sum is over all p such that $\varphi(p) = q$, and k_p denotes the ramification index of φ at p.

To reduce notation somewhat, let us define the function $D\colon Y \longrightarrow \mathbb{N}$ by $D(q) = \sum_{p\in\varphi^{-1}(q)} k_p$. The goal of this problem is then to show that D is a constant function.

LEMMA : For each $q \in Y$ there is a small neighbourhood (= open set around) V of q such that D is constant on V.

First let us see how to prove the result using the lemma.

(a) Use the lemma to show that for each $d \in \mathbb{N}$ the set

$$D^{-1}(d) = \left\{ q \in Y \,\middle|\, D(q) = d \right\}$$

is open.

(b) Use (a) to show that for each $d \in \mathbb{N}$ the set $D^{-1}(d)$ is closed. (SUGGESTION: this is the same as showing that the complement is open.)

(c) Use (a)+(b) to show that for each $d \in \mathbb{N}$, $D^{-1}(d)$ is either Y or the empty set.

(d) Conclude that there is a unique $d \in \mathbb{N}$ such that $D^{-1}(d) = Y$, i.e., conclude that D is constant on Y.

We now work on proving the lemma.

Fix $q \in Y$, and suppose that $\varphi^{-1}(q) = \{p_1, p_2, \ldots, p_r\}$. From our local picture we know that there is an open set V around q, and open sets $U_1, \ldots, U_r$ around $p_1, \ldots, p_r$ such that $\varphi(U_i) \subset V$ for each $i = 1, \ldots, r$, and that on each U_i the map φ looks like $z_i \mapsto z_i^{k_{p_i}}$, where z_i is a local coordinate on U_i, and k_{p_i} the ramification index at p_i.

Given these U_i and V, for $q \in V$ let us split our function D into the sum of two functions. For $q' \in V$, by definition $D(q')$ is the sum over $p' \in \varphi^{-1}(q')$ of the ramification indices $k_{p'}$. We will split the sum into pieces according to whether p' is in $U_1 \cup U_2 \cup \cdots \cup U_r$ or outside it. Set $U = U_1 \cup U_2 \cup \cdots \cup U_r$ and define :

$$D_U(q') = \sum_{p'\in\varphi^{-1}(q')\cap U} k_{p'} \text{ and } D_U^c \sum_{p'\in\varphi^{-1}(q'), p\notin U} k_{p'},$$

so that $D(q') = D_U(q') + D_U^c(q')$. (The "$c$" is for "complement.)

Set $d = D(q) = k_{p_1} + k_{p_2} + \cdots + k_{p_r}$.

(e) Show that for q' sufficiently close to q, $D_U(q') = d$.

Claim: For q' sufficiently close to q, all points of $\varphi^{-1}(q')$ are in U. (This then shows that for those points $D_U^c(q') = 0$, and hence using $D = D_U + D_U^c$ and (e) that $D(q') = d$ for all points q' sufficiently close to q, thus proving the lemma.)

The negation of this claim is that there is a sequence of points q_1', q_2', ..., converging to q, and for each q_i' a point $p_i' \in \varphi^{-1}(q_i')$ which is outside of U. Since X is compact, such a sequence would have a limit point $\overline{p} \in X$.

(f) Explain why we would have $\varphi(\overline{p}) = q$.

(g) Explain why this means that $\overline{p} \in \{p_1, p_2, \ldots, p_r\}$.

(h) Explain why this means that some p_i' (in fact, infinitely many p_i') would have to be in U.

(i) Explain why this is a contradiction, thus establishing the claim, the lemma, and finally the theorem from class.