1. Let A be a ring (i.e., a commutative ring) which is a domain and has finitely many elements. In this problem we will show that A is a field. Let $a \in A$, $a \neq 0$ be an element.

(a) Consider the map $\varphi_a \colon A \longrightarrow A$ given by multiplying by a (i.e, $\varphi_a(b) = ab$ for all $b \in A$), and show that this map is injective.

(b) Since A is finite, explain why φ_a must also be surjective.

(c) Explain why there must be an element $b \in A$ such that $ab = 1$.

(d) Explain why A is a field.

2. Let $K \subseteq L$ be fields, and S_1 and S_2 two subsets of L. If we adjoin S_1 to K we get the field $K(S_1)$, and we could then adjoin S_2 to get the field $(K(S_1))(S_2)$. Show that this field is the same as $K(S_1 \cup S_2)$, obtained by adjoining the union of S_1 and S_2.

SUGGESTION: Use the defining properties of "field obtained by adjoining elements" to show that each of the fields is contained in the other.

3. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. (HINT: One inclusion should be obvious, and the other should follow after a little algebra.)

4. In our argument that $\left\{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\right\}$ is a field we needed to use the identity

$$(a + b\sqrt[3]{2} + c\sqrt[3]{4}) \cdot \Big((a^2 - 2bc) + (2c^2 - ab)\sqrt[3]{2} + (b^2 - ac)\sqrt[3]{4}\Big) = a^3 + 2b^3 + 4c^3 - 6abc$$

to "get the cube roots out of the denominator". There is a gap in this argument not addressed in class : if a, b, and c are such that $a + b\sqrt[3]{2} + c\sqrt[3]{4} \neq 0$, how do we know that $a^3 + 2b^3 + 4c^3 - 6abc \neq 0$? (That's something we can't allow in a denominator.)

In this question we will justify that assertion, although we will assume something that we haven't proven yet : that 1, $\sqrt[3]{2}$ and $\sqrt[3]{4}$ are linearly independent over $\mathbb{Q}$. You may assume this for the question.

Let $\gamma = a + b\sqrt[3]{2} + c\sqrt[3]{4}$ be an element of $\mathbb{Q}(\sqrt[3]{2})$, with $a, b, c \in \mathbb{Q}$, and consider the map $\varphi \colon \mathbb{Q}(\sqrt[3]{2}) \longrightarrow \mathbb{Q}(\sqrt[3]{2})$ given by multiplication by γ.

(a) Prove that φ is a $\mathbb{Q}$-linear map.

(b) Write out the matrix for this map in the $\mathbb{Q}$-basis $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$.

(c) Compute the determinant of this matrix.

(d) If $\gamma \neq 0$, explain why $a^3 + 2b^3 + 4c^3 - 6abc \neq 0$.

NOTE: We will soon have a different way of showing that the set $\left\{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\right\}$ is a field, without needing the identity above, and without needing to prove that $a^3 + 2b^3 + 4c^3 - 6abc \neq 0$ whenever $\gamma \neq 0$. The computation is still useful however, and we will come back to the meaning of the determinant later in the course.