Math 414 Homework Assignment 2

DUE DATE: JAN. 21, 2016

1. Let $f(x) = x^3 + 3x^2 + 3x - 1 \in \mathbb{Q}[x]$.

(a) Find the remainder of x^4 when divided by $f(x)$.

(b) Find the reminder of $(x^2 + 1)^3$ when divided by $f(x)$.

(c) Find polynomials $u(x)$, $v(x) \in \mathbb{Q}[x]$, with $\deg(u(x)) \leqslant 2$ which solve

$$x^2 \cdot u(x) + v(x)f(x) = 1.$$

2. Let α be the real number $\alpha = 2^{1/3} - 1$. To as many decimal places as you can (well, at least 8, and no more than 20), evaluate the following real numbers:

(a) α^4;

(b) $(\alpha^2 + 1)^3$;

(c) $1/\alpha^2$;

(d) $3\alpha^2 + 10\alpha + 12$;

(e) $24\alpha^2 + 60\alpha - 16$;

(f) $6\alpha^2 + 10\alpha - 3$.

Now,

(g) explain why some of the numbers this question were the same (question 1 may help).

3. In this question we will show that $f(x) = x^4 - 10x^2 + 1$ is the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$. Let $q(x) \in \mathbb{Q}[x]$ be the (at the moment unknown) minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$. It is easy to check that $f(\sqrt{2} + \sqrt{3}) = 0$, which implies that $q(x) \mid f(x)$. To show that $q(x) = f(x)$ we may therefore show either that $f(x)$ is irreducible in $\mathbb{Q}[x]$ or that $\deg(q(x)) = 4$.

We will use equality $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$, proved in the last homework assignment to show that $\deg(q(x)) = 4$.

(a) Using the chain of field extensions $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$ explain why $\deg(q(x))$ must be even.

Since $\deg(q(x)) \leqslant 4$, this means that we must have $\deg(q(x)) = 2$ or 4. We now assume that $\deg(q(x)) = 2$ and show how this leads to a contradiction.

(b) Explain why $\deg(q(x)) = 2$ implies that $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$, and similarly that $\mathbb{Q}(\sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

(c) Part (b) gives us $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{3})$, and if so we would be able to write $\sqrt{3} = a+b\sqrt{2}$ with $a, b \in \mathbb{Q}$. Square both sides and show how this would lead to a contradiction. (Do not forget to deal with the special cases $a = 0$ or $b = 0$.)

Thus (after finishing (c)) we conclude that $f(x)$ is the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$. Let us also try the other method of showing that $f(x)$ is the minimal polynomial : showing that $f(x)$ is irreducible over $\mathbb{Q}$.

(d) Use one of the irreducibility tests from class to show that $f(x)$ is irreducible over $\mathbb{Q}$. (There is more than one that will work.)

4. In this question we will explore some aspects of numbers algebraic over a fixed field.

(a) Suppose that $K \subseteq M$ is a field extension, with $[M : K] = d$ (in particular, the degree of the extension is finite). Show that every $\alpha \in M$ is algebraic over K, and satisfies a polynomial of degree $\leqslant d$. (Suggestion: Can 1, α,..., α^d be linearly independent over K?)

(b) Let $K \subseteq L$ be a field extension, and $\alpha, \beta \in L$. If β is algebraic over K, show that β is algebraic over $K(\alpha)$.

(c) If $\alpha, \beta \in L$ are both algebraic over K, show that $[K(\alpha, \beta) : K]$ is finite.

(d) If α, $\beta \in L$ are algebraic over K with $\beta \neq 0$, show that $\alpha+\beta$, $\alpha\beta$, and α/β are algebraic over K

(e) Consider the set $\overline{\mathbb{Q}} = \left\{\alpha \in \mathbb{C} \,\middle|\, \alpha \text{ is algebraic over } \mathbb{Q}\right\}$. Show that $\overline{\mathbb{Q}}$ is a field.

(f) Are there irreducible polynomials in $\mathbb{Q}[x]$ of arbitrarily large degree?

(g) Is $[\overline{\mathbb{Q}} : \mathbb{Q}]$ finite or infinite?

(h) Does the converse to (a) hold? I.e., if $K \subseteq M$ is a field extension such that every $\alpha \in M$ is algebraic over K, does this imply that $[M : K]$ is finite?