DUE DATE: FEB. 4, 2016

1. Let $K \subseteq L$ be fields, and $\alpha \in L$. We understand the structure of $K(\alpha)$ when α is algebraic over K. In this question we will deal with the case that α is transcendental over K.

Suppose that α is transcendental over K and let $\varphi_\alpha\colon K[x] \longrightarrow L$ be the evaluation map sending $f(x) \in K[x]$ to $f(\alpha) \in L$. Recall that this is a ring homomorphism.

(a) Explain why φ_α is injective.

(b) Explain how to use φ_α to get a homomorphism of fields $K(x) \longrightarrow L$. (SUGGESTION: It is just like our argument that $\mathbb{Q}$ is a subfield of every field of characteristic 0, starting from the point where we know that $\mathbb{Z}$ is a subring of every such field.)

(c) Prove that $K(\alpha) \cong K(x)$.

(d) Are $\mathbb{Q}(\pi)$ and $\mathbb{Q}(e)$ isomorphic fields? (Here $\pi \cong 3.14159265\ldots$ and $e \cong 2.718281828\ldots$ are the usual numbers we know.)

2. Let p be a prime, n a positive integer, and write $n = mp^k$ with $p \nmid m$. For any a, $0 \leqslant a \leqslant n$, prove that

$$\binom{n}{a} \equiv \begin{cases} 0 & \bmod p \quad \text{if } p^k \nmid a \\ \binom{m}{\frac{a}{p^k}} & \bmod p \quad \text{if } p^k \mid a. \end{cases}$$

(SUGGESTION: Consider $(x+1)^n \bmod p$, i.e, in $\mathbb{F}_p$.)

3. Let K be a field with $\mathrm{Char}(K) \neq 2$ and suppose that L/K is a degree 2 extension. By the argument in class, that means we can express L as $K(\sqrt{\gamma})$ for some $\gamma \in K$. In class we showed that L/K must be a normal extension.

(a) Show that L/K is also a separable extension.

(b) Compute $\mathrm{Aut}(L/K)$ and describe how each element of the group acts on L.

4. In class we saw that if K is field and $q(x) \in K[x]$ an irreducible polynomial such that $q'(x) \neq 0$, then $q(x)$ had no repeated roots. Prove a slightly more general version of this result : show that $f(x) \in K[x]$ has no repeated roots if and only if $\gcd(f(x), f'(x)) = 1$.

5. For each of the following polynomials f_i, let L_i be the field generated by $\mathbb{Q}$ and all the roots of f_i. That is, if $\alpha_1,\ldots,\ \alpha_r$ are the roots of f_i, let $L_i = \mathbb{Q}(\alpha_1,\ldots,\alpha_s)$. (In other words, L_i is the *splitting field* of each f_i.) In each case find all the roots of f_i, and find the degree of L_i over $\mathbb{Q}$.

(a) $f_1 = x^4 - 5x^2 + 6$.

(b) $f_2 = x^3 - 1$.

(c) $f_3 = x^6 - 1$.

(d) $f_4 = x^6 - 2$.