Math 414 — Homework Assignment 6

DUE DATE: FEB. 25, 2016

1. Let L/K be a finite extension and $G = \mathrm{Aut}(L/K)$. Even if L/K is not a Galois extension we always have order-reversing maps of lattices

$$\begin{array}{ccc} & H \longmapsto L^H & \\ \Big\{\text{lattice of subgroups } H \text{ of } G\Big\} & \overrightarrow{\overleftarrow{\qquad\qquad}} & \Big\{\text{lattice of intermediate fields } M\Big\} \\ & \mathrm{Aut}(L/M) \longleftarrow\!\!\dashv M & \end{array}$$

However, if L/K is not a Galois extension, there is no reason that these maps have to be bijections. In this problem we will see this in a very simple example. (In some sense the example may be too small to be convincing, but it does show that the correspondence doesn't work out in general.)

Let $L = \mathbb{Q}(\sqrt[3]{2})$ and $K = \mathbb{Q}$.

(a) Is L/K a Galois extension?

(b) Find $[L : K]$.

(c) Find all intermediate fields M, $K \subseteq M \subseteq L$. (SUGGESTION: Consider the tower law $[L : K] = [L : M] \cdot [M : K]$ and find the possible degrees of the intermediate fields first.)

(d) Write down the lattice of intermediate fields.

(e) Let $G = \mathrm{Aut}(L/K)$. If $\sigma \in G$ explain where σ must send $\sqrt[3]{2}$. (SUGGESTION: As usual you should start with the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$.)

(f) Compute G (i.e., find all elements of G).

(g) Write down the lattice of all subgroups of G. (This will be quite small.)

(h) For each subgroup H of G, find L^H.

(i) For each intermediate field M, find $\mathrm{Aut}(L/M)$.

2. Suppose that (α_1, β_1), ..., (α_k, β_k) are points of $\mathbb{C}^2$ (i.e, $\alpha_i, \beta_i \in \mathbb{C}$), and that the set $S = \{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$ is stable under complex conjugation. (This means that if $(\alpha_i, \beta_i) \in S$ then $(\overline{\alpha_i}, \overline{\beta_i}) \in S$ too). For any $d \geqslant 0$, consider the $\mathbb{C}$-vector space V_d of polynomials of degree $\leqslant d$ in $\mathbb{C}[x, y]$ which are zero at all (α_i, β_i), $i = 1, \ldots, k$. Show that V_d has a basis consisting of polynomials with real coefficients.

3. In this problem we will work out the Galois correspondence in the case $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $K = \mathbb{Q}$. Recall that from **H3** Q2(d) we know that $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is a basis of L/K.

(a) Show that L/K is a Galois extension.

Let $G = \mathrm{Gal}(L/K)$. In this case it turns out that G is the Klein four-group, $G = \{e, \tau_1, \tau_2, \tau_1\tau_2\}$ where all elements except e have order 2, and τ_1 and τ_2 commute. The action of G on L may be deduced from the information :

$$\boxed{\tau_1:\quad \begin{array}{l} \sqrt{2} \longmapsto -\sqrt{2} \\ \sqrt{3} \longmapsto \sqrt{3} \end{array}} \quad \text{and} \quad \boxed{\tau_2:\quad \begin{array}{l} \sqrt{2} \longmapsto \sqrt{2} \\ \sqrt{3} \longmapsto -\sqrt{3} \end{array}} .$$

(b) Deduce the action of τ_1, τ_2 on $\sqrt{6}$.

(c) Deduce the action of τ_1, τ_2, and $\tau_1\tau_2$ on an arbitrary element $a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}$ of L (with $a, b, c, d \in \mathbb{Q}$).

(d) Find all subgroups of G and write down the (reversed) lattice of subgroups of G

(e) For each subgroup H of G, find the fixed field L^H.

Suggestion: To find the elements of L fixed by an element σ of G, start with a general element $\alpha = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ of L, write down the equation $\sigma(\alpha) = \alpha$, and consider it as a system of linear equations in the unknowns a, b, c, and d. Solutions to the equations are elements of L fixed by σ. (Here you will need to use your formula from (c) to see what $\sigma(\alpha)$ is.)

(f) Write down the lattice of intermediate fields of L/K.

4. Let L/K be a Galois extension, $G = \mathrm{Gal}(L/K)$, and set $d = |G| = [L : K]$. Let $\sigma_1, \ldots, \sigma_d$ be the elements of G, and choose any basis $\alpha_1, \ldots, \alpha_d$ of L over K. Explain why the determinant

$$\begin{vmatrix} \sigma_1(\alpha_1) & \sigma_1(\alpha_2) & \sigma_1(\alpha_3) & \cdots & \sigma_1(\alpha_d) \\ \sigma_2(\alpha_1) & \sigma_2(\alpha_2) & \sigma_2(\alpha_3) & \cdots & \sigma_2(\alpha_d) \\ \sigma_3(\alpha_1) & \sigma_3(\alpha_2) & \sigma_3(\alpha_3) & \cdots & \sigma_3(\alpha_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_d(\alpha_1) & \sigma_d(\alpha_2) & \sigma_d(\alpha_3) & \cdots & \sigma_d(\alpha_d) \end{vmatrix} \neq 0.$$

(Suggestion : Consider the matrix as giving a linear map $L^d \longrightarrow L^d$ and use part of the argument from the proof of Artin's lemma.)