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1. NEWTON'S RECURSION FOR THE POWER SUMS. Fix $n \geqslant 1$ and variables $x_1, \ldots,$ x_n. For any $m \geqslant 1$ the m-th power sum is $P_m = x_1^m + x_2^m + \cdots + x_n^m$. Since P_m is a symmetric polynomial Newton's theorem tells us we may express P_m in terms of the elementary symmetric polynomials $e_1, \ldots, e_n$. A recursive formula, also due to Newton, gives a quick way to do this.

For purposes of the recursion we will use the symbol e_r for all $r \geqslant 1$, with the convention that $e_r = 0$ if $r > n$. We also set $P_0 = 1$, contrary to the formula for $m \geqslant 1$. Newton's recursion is

$$P_m = e_1 P_{m-1} - e_2 P_{m-2} + e_3 P_{m-3} - \cdots - (-1)^{m-1} e_{m-1} P_1 - (-1)^m m e_m P_0.$$

(Don't miss the factor of m in the final term.) Starting with $P_1 = e_1$, this tells us how to express the power sums in terms of the elementary symmetric polynomials. E.g., when $n = 2$ we have

$$P_1 = e_1; \quad P_2 = e_1 P_1 - 2e_2 P_0 = e_1^2 - 2e_2; \quad P_3 = e_1 P_2 - e_2 P_1 + 3e_3 P_0 = e_1^3 - 3e_1 e_2.$$

Note that in computing P_3 we have used our convention that $e_3 = 0$ (since $n = 2$).

(a) Suppose that $n = 3$. Use Newton's recursion to compute the formulae for P_2, P_3, and P_4.

(b) Let α_1, α_2, and α_3 be the roots of $f = x^3 + 5x^2 + 6x - 4$. Compute $\alpha_1^4 + \alpha_2^4 + \alpha_3^4$.

2. For each of the following cubic polynomials f compute $\mathrm{Gal}(L/\mathbb{Q})$ where L is the splitting field of f. If you are claiming that the polynomial f is irreducible over $\mathbb{Q}$, be sure to include a justification. (CAUTION: at least one of the cubics is reducible.) Also recall that if $f = x^3 + bx^2 + cx + d$, then

$$\Delta(f) = b^2c^2 + 18bcd - 4b^3d - 4c^3 - 27d^2.$$

(a) $f_1 = x^3 + 4x^2 + x + 1$.

(b) $f_2 = x^3 + 4x^2 + 2x - 2$.

(c) $f_3 = x^3 + 4x^2 + 3x - 1$.

(d) $f_4 = x^3 + 4x^2 + 4x + 1$.

3. In this problem we will see what the sign of the discriminant of a real cubic polynomial tells us about its real or complex roots. Let $f = x^3 + bx^2 + cx + d \in \mathbb{R}[x]$. We assume that f has distinct roots, that is, that $\Delta(f) \neq 0$. We do not need to assume that f is irreducible.

(a) Explain why f has to have at least one real root. (HINT: This is really a problem in calculus, in particular, the intermediate value theorem may be useful.)

Let α_1 be any real root, and α_2 and α_3 the other two roots.

(b) Assume that α_2 and α_3 are real. Show that $\Delta(f) > 0$. (SUGGESTION: first show that $\delta(f) \in \mathbb{R}$.)

(c) Now assume that α_2 is not real. Show that $\alpha_3 = \overline{\alpha}_2$, i.e., that α_2 and α_3 are conjugate complex numbers.

(d) By (c) we may write $\alpha_2 = a - bi$ and $\alpha_3 = a + bi$ for some $a, b \in \mathbb{R}$, with $b \neq 0$. Show that $\Delta(f) < 0$. (SUGGESTION: first show that $\delta(f)$ is purely imaginary, i.e., of the form $i \cdot t$ for some real number $t \neq 0$.)

4. In this problem we will work out a few other identities involving the discriminant. Let $f \in K[x]$ be a monic polynomial of degree n, with roots $\alpha_1, \ldots, \alpha_n$.

(a) For any $c \in K$, show that $\Delta(f(x + c)) = \Delta(f(x))$. That is, show that translating the polynomial does not change the discriminant. (E.g. for $f = x^3 + 5x^2 + 3x + 1$, $f(x - 2) = (x - 2)^3 + 5(x - 2)^2 + 3(x - 2) + 1 = x^3 - x^2 - 5x + 7$ has the same discriminant as f.) SUGGESTION: What are the roots of $f(x + c)$?

(b) For any number β, explain why $(\beta - \alpha_1)(\beta - \alpha_2) \cdots (\beta - \alpha_n) = f(\beta)$.

(c) Let $g \in K[x]$ be a monic polynomial of degree m with roots $\beta_1, \ldots, \beta_m$. Show that $\left(\prod_{i=1}^{m} f(\beta_i)\right)^2 = \left(\prod_{j=1}^{n} g(\alpha_j)\right)^2$

(d) Show that $\Delta(f \cdot g) = \Delta(f) \cdot \Delta(g) \cdot \left(\prod_{i=1}^{m} f(\beta_i)\right)^2$.