Math 414 Homework Assignment 11

DUE DATE: MAR. 31, 2016

1. In this problem we will investigate solvability for a particular matrix group. For a general group G, we say that G is *solvable* if G has a composition series

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G$$

where each factor G_i/G_{i-1} is an abelian group. When G is a finite group our lemma from class shows this is the same as requiring that G has a composition series where each factor is cyclic of prime order. For infinite groups there is no guarantee that there are quotients of finite order, and it turns out that having factors which are abelian is still a useful notion, which is why the definition above is the general one.

Let B be the *Borel subgroup* of $\mathrm{GL}_2(\mathbb{R})$, i.e., B is the subgroup

$$B = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \,\Big|\, a, b, d \in \mathbb{R},\ ad \neq 0 \right\}$$

of upper triangular matrices. Consider the following subgroups of B :

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subset \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \,\Big|\, b \in \mathbb{R} \right\} \subset \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \,\Big|\, a, b \in \mathbb{R},\ a \neq 0 \right\} \subset B$$

Show that each subgroup is normal in the next (i.e, that they form a composition series) and that each quotient is abelian. Also, identify each quotient group (the quotients are abelian groups which you should know, or at least be able to describe).

2. In this problem we will prove part (b) of the lemma on radical extensions from class. That is, we will show that if M is a field of characteristic zero which contains a primitive m-th root of unity ζ, and β and element such that $\beta^m \in M$, then the extension $M(\beta)/M$ is Galois with abelian Galois group.

Set $\gamma = \beta^m$. Then β is a root of $f(x) = x^m - \gamma \in M[x]$.

(a) Find the roots of f and explain why they are all in $M(\beta)$. (HINT: Don't forget that $\zeta \in M$.)

(b) Let $q(x)$ be the minimal polynomial polynomial of β over M. Explain why $q(x) \mid f(x)$.

(c) Explain why $q(x)$ splits completely in $M(\beta)$.

(d) Explain why $M(\beta)/M$ is a Galois extension.

Let $G = \mathrm{Gal}(M(\beta)/M)$. Since β is a generator of $M(\beta)/M$, to understand how $\sigma \in G$ acts on $M(\beta)$ it is enough to understand what σ does to β.

(e) Explain why $\sigma(\beta) = \beta \cdot \zeta^n$ for some $n \in \{0, \ldots, m-1\}$.

Let us use σ_n to name an element such that $\sigma_n(\beta) = \beta \cdot \zeta^n$. (So $\sigma_3(\beta) = \beta \cdot \zeta^3$, $\sigma_4(\beta) = \beta \cdot \zeta^4$, $\sigma_0(\beta) = \beta \cdot \zeta^0 = \beta$, etc.) Note we are not claiming that $\sigma_n \in G$ for every possible n, just that this gives a consistent way of giving a name to the elements of G.

(f) Suppose that $\sigma_{n_1}, \sigma_{n_2} \in G$. Show that $\sigma_{n_1}\sigma_{n_2} = \sigma_{n_2}\sigma_{n_1}$ by computing what each side does to β. (HINT: ζ is in M.)

Since σ_{n_1} and σ_{n_2} were arbitrary, this means that G is abelian, proving this part of the lemma.

3. In this problem we will see what the general constructions from class mean in two examples we have already computed. In each of the examples we had $K = \mathbb{Q}$, and L the splitting field of a polynomial of the form $f(x) = x^m - \gamma$, with $\gamma \in K$ (i.e, in $\gamma \in \mathbb{Q}$).

(a) On February 22nd and 24th ("An example", and "An example continued") we computed the Galois group of the splitting field of $f(x) = x^3 - 2$. (The answer was that the Galois group is $S_3 = D_3$.) Let L be this splitting field, i.e., $L = \mathbb{Q}(2^{\frac{1}{3}}, \omega)$ where $\omega = e^{\frac{2\pi i}{3}}$. The tower of fields

$$\mathbb{Q} \subset \mathbb{Q}(\omega) \subset \mathbb{Q}(\omega, 2^{\frac{1}{3}}) = L$$

is a radical tower. Identify the subgroups of $\mathrm{Gal}(L/K)$ corresponding to each of the fields in the tower, and compute the factors of the resulting composition series. (By our lemma from class, they should all be abelian.)

(b) On February 26th ("A more complicated example") we computed the Galois group of the splitting field of $f(x) = x^4 - 5$. (The answer was that the Galois group is D_4.) Let $L = \mathbb{Q}(5^{\frac{1}{4}}, i)$ be the splitting field. The tower of fields

$$\mathbb{Q} \subset \mathbb{Q}(i) \subset \mathbb{Q}(i, 5^{\frac{1}{4}}) = L$$

is a radical tower. Identify the subgroups of $\mathrm{Gal}(L/K)$ belonging to each of the fields in the tower, and compute the factors of the resulting composition series. (They should again all be abelian.)

In this question you can freely use the details we computed in class (e.g., the Galois correspondence) — there is no need to work it out again from scratch.