Instructions: The exam has five sets of questions labeled I through V. You should do all five questions.

You may freely use any notes or statements from class or the book. It is also permissible to use outside sources but if you do this be sure to acknowledge these references in your write up of those questions. (You can also come and ask me questions if you get stuck).

The exam is due at **Friday, April 15, 2016 at 13:00**.
(Of course you may hand it in earlier if you wish.)

Questions:

I. — **ADJOINT FUNCTORS.**
II. — **INTEGRAL CLOSURES.**
III. — **RIGHT EXACTNESS.**
IV. — **INTEGRALITY OF CHARACTERS.**
V. — **QUESTIONS ABOUT Tor.**
I. — ADJOINT FUNCTORS

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F: \mathcal{C} \rightarrow \mathcal{D}$ a functor which has a left adjoint.

Prove that $F$ preserves fibre products. This means that, given any three objects $X$, $Y$, and $Z$ of $\mathcal{C}$, with morphisms $\pi_1: X \rightarrow Z$ and $\pi_2: Y \rightarrow Z$, prove that if $X \times_Z Y$ exists in $\mathcal{C}$ then $F(X) \times_{F(Z)} F(Y)$ exists in $\mathcal{D}$ and that $F(X) \times_{F(Z)} F(Y) = F(X \times_Z Y)$.

II. — INTEGRAL CLOSURES

Let $A$ be a ring. Let $a_0, \ldots, a_{n-1}$ be elements of $A$, and let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$. Set $M = A[x]/(p(x))$; this is a free $A$-module of rank $n$.

(a) Let $\varphi_x: M \rightarrow M$ be the $A$-module homomorphism which is “multiplication by $x$”, i.e., by the image of $x$ under the quotient map $A[x] \rightarrow M$. Find the characteristic polynomial of $\varphi_x$.

Now suppose that $A$ is a domain, with quotient field $K$, and $L$ any extension field of $K$. An element $\alpha$ of $L$ is called integral over $A$ if there is a monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ in $A[x]$ with $\alpha$ as a root. (Monic means that the leading coefficient of the polynomial, the coefficient of $x^n$, is 1.)

(b) Prove that the following are equivalent:

(b1) The element $\alpha \in L$ is integral over $A$.

(b2) There is a free $A$-module $M$ and an $A$-module map $\varphi: M \rightarrow M$ such that $\alpha$ is an eigenvalue of $\varphi \otimes \text{Id}_L : M \otimes_A L \rightarrow M \otimes_A L$. (Note: $M \otimes_A L$ is a finite dimensional vector space over $L$, and $\text{Id}_L$ is the identity map on $L$).

(c) Suppose that $\alpha$ and $\beta$ are elements of $L$ which are integral over $A$. Let $(M, \varphi)$, and $(N, \psi)$ be the free modules and $A$-module maps guaranteed by (b2) above. Considering the $A$-module maps $\varphi \otimes \psi$ and $\varphi \otimes \text{Id}_N + \text{Id}_M \otimes \psi$ from $M \otimes_A N$ to itself, prove that $\alpha \beta$ and $\alpha + \beta$ are integral over $A$.

(d) The set of elements $B$ in $L$ which are integral over $A$ is called the integral closure of $A$ in $L$. Prove $B$ is a subring of $L$ and that $A \subseteq B$.

A domain $A$ is called integrally closed if the integral closure of $A$ in its own quotient field $K$ is $A$ itself.

(e) Prove that the ring $A = \mathbb{Z}[\sqrt{5}]$ is not integrally closed.

(f) Prove that $A = \mathbb{Z}$ is integrally closed.
III. — RIGHT EXACTNESS

Let $A$ be a ring.

(a) Say what it means for a functor $F : A \to A \to A \to A$ to be right exact.

(b) Is $\text{Sym}^n : A \to A \to A \to A$ right exact for $n \geq 2$?

(c) Is $\Lambda^n : A \to A \to A \to A$ right exact for $n \geq 2$?

IV. — INTEGRALITY OF CHARACTERS

Let $G$ be a finite group of order $n$. In this question all representations will be over $\mathbb{C}$.

(a) Let $\rho : G \to \text{GL}(V)$ be a representation of $G$. Explain why $\rho(g)^n = \text{Id}_V$ for all $g \in G$.

(b) Show that all eigenvalues of $\rho(g)$ are $n$-th roots of unity.

(c) Let $B$ be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\zeta_n)$, where $\zeta_n = e^{2\pi i/n}$. Show that $\chi_{\rho}(g) \in B$ for all $g \in G$.
   (Question II is relevant.)

(d) Suppose that $\chi_{\rho}(g) \in \mathbb{Q}$. Show that $\chi_{\rho}(g) \in \mathbb{Z}$. (See II again.)

V. — QUESTIONS ABOUT $\text{Tor}$

Let $A$ be any ring.

(a) Show that for any $A$-modules $M, N_1$, and $N_2$, that $\text{Tor}_i(M, N_1 \oplus N_2) = \text{Tor}_i(M, N_1) \oplus \text{Tor}_i(M, N_2)$ for all $i \geq 0$.

Now let $k$ be a field, $A = k[x, y]$, and set $M = A/(x, y) \cong k$, where $(x, y)$ is the ideal generated by $x$ and $y$.

(b) How do $x$ and $y$ act on $M$?

(c) Show that the complex below is a projective resolution of $M$:

$$
\begin{array}{cccc}
A \xrightarrow{\delta_2} A \oplus A \xrightarrow{\delta_1} A & \xrightarrow{\delta_3} & M \\
\downarrow{h} & & \downarrow & \\
(hy, -hx) & & (f, g) & \mapsto fx + gy
\end{array}
$$

(d) Compute $\text{Tor}_i(M, M)$ for all $i \geq 0$.

(e) Let $N = A/(y)$. Compute $\text{Tor}_i(M, N)$ for all $i \geq 0$. 