1. Let $C$ be a category, $Z$ an object of $C$, and let $D$ be the category of objects over $Z$ (i.e., the category denoted $C/Z$ in class).

(a) Prove that product in $D$ is the fibre product in $C$. I.e., given two ‘objects’ in $D$, prove that their product in $D$ (if it exists) is the same as the fibre product (if it exists) corresponding to those objects in $C$.

There are a few details missing in this description (for instance, what are objects and morphisms in $D$?) and you should fill them in in your answer.

(b) Give an example of a category $C$ where products don’t always exist, i.e., for which there exist two objects $X, Y$ of $C$ such that $X \times Y$ does not exist in $C$.

2. Let $i : \text{Ab} \hookrightarrow \text{Grp}$ be the inclusion functor from abelian groups to groups (i.e., for any abelian group $G$, $i(G)$ is the same group but now considered in the category of all groups). Let $\pi : \text{Grp} \longrightarrow \text{Ab}$ be the functor $G \mapsto G/[G,G]$ from Homework 1, Question 2.

(a) Prove that $i$ and $\pi$ are an adjoint pair.

This includes figuring out which one is the left and which one is the right adjoint in the pair, describing explicitly how you get an isomorphism between the Hom’s, as well as checking the naturality conditions. Feel free to rename one functor $F$, and one functor $G$ if that helps, although without font assistance it might conflict with the $G$’s you use to indicate a group.

(b) Suppose that $G_1$ and $G_2$ are groups and let $G = G_1 \ast G_2$ be their free product. If for any group $H$, $H'$ denotes $H/[H,H]$ (so $G' = G/[G,G], G'_1 = G_1/[G_1,G_1], etc$) prove that $G' = G'_1 \oplus G'_2$. 
3. Let $k$ be a field, and $I$ an index set, possibly infinite. Suppose that, for each $i \in I$, $V_i$ is a vector space over $k$, and let $W$ also be a vector space over $k$. (Note: Vector spaces aren’t necessarily finite dimensional.)

In the category $\mathbf{Vect}_k$ of vector spaces over $k$, prove that

(a) $\text{Hom}(\bigoplus_{i \in I} V_i, W) \cong \prod_{i \in I} \text{Hom}(V_i, W)$, and

(b) $\text{Hom}(W, \prod_{i \in I} V_i) \cong \prod_{i \in I} \text{Hom}(W, V_i)$.

Give examples to show why these other six possible combinations of $\text{Hom}$, $\bigoplus$, and $\prod$ are not necessarily isomorphic (i.e., prove that the following aren’t always isomorphic)

(c) $\text{Hom}(\bigoplus_{i \in I} V_i, W) \not\cong \bigoplus_{i \in I} \text{Hom}(V_i, W)$,

(d) $\text{Hom}(W, \prod_{i \in I} V_i) \not\cong \bigoplus_{i \in I} \text{Hom}(W, V_i)$,

(e) $\text{Hom}(\prod_{i \in I} V_i, W) \not\cong \prod_{i \in I} \text{Hom}(V_i, W)$,

(f) $\text{Hom}(W, \bigoplus_{i \in I} V_i) \not\cong \prod_{i \in I} \text{Hom}(W, V_i)$,

(g) $\text{Hom}(\prod_{i \in I} V_i, W) \not\cong \bigoplus_{i \in I} \text{Hom}(V_i, W)$, and

(h) $\text{Hom}(W, \bigoplus_{i \in I} V_i) \not\cong \bigoplus_{i \in I} \text{Hom}(W, V_i)$.

Note: After thinking about this question for a bit, you may notice that the second part is not formulated very precisely. Part of the question is grappling with this, and figuring out what the question might/could/should mean.

This part of the problem is meant to replicate an experience you may have reading a mathematical paper. Sometimes equalities or isomorphisms (or non-isomorphisms) are claimed, without much explanation. Are they true? Did the author have any additional ‘naturality’ conditions in mind when making the claim? You, as the reader, will have to struggle to sort out the author’s hidden intentions, and the possible truth of the statement.