1. Let n be any positive integer. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$. Deduce that there are no non-trivial bilinear maps from $\mathbb{Q} \times (\mathbb{Z}/n\mathbb{Z})$ to any abelian group G.

2. Suppose that V and W are vector spaces over a field k. An element α of the tensor product $V \otimes_k W$ is called *pure* if it is of the form $v \otimes w$ for vectors v in V and w in W.

A general element $\alpha \in V \otimes W$ is a sum of pure elements.

Suppose that V is an r-dimensional vector space with basis $e_1, \ldots, e_r$ and that W is an s-dimensional vector space with basis $f_1, \ldots, f_s$. We know that $\{e_i \otimes f_j\}_{i=1,j=1}^{r,s}$ is a basis for $V \otimes_k W$ so that any $\alpha \in V \otimes_k W$ can be written uniquely as $\alpha = \sum_{i,j} c_{ij} e_i \otimes f_j$.

(a) Prove that an element $\alpha \in V \otimes_k W$ is pure if and only if the matrix $[c_{ij}]$ for α has rank 1.

(b) Suppose that $r = s = 2$. Find an element of $V \otimes_k W$ which isn't pure. (i.e., show that a general element of $V \otimes_k W$ is really a nontrivial sum of pure elements, so that the pure elements, which we often talk about when defining maps, aren't everything).

(c) In the general case (of arbitrary r and s) show that a general element of $V \otimes_k W$ is the sum of at most $\min(r, s)$ pure elements.

3. Let k be a field, V a vector space of dimension r over k and W a vector space of dimension s over k (both r and s are finite). Suppose that $\varphi\colon V \longrightarrow V$ and $\psi\colon W \longrightarrow W$ are linear transformations. You may assume that k is algebraically closed.

(a) If both φ and ψ are diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_r$ and $\mu_1, \ldots, \mu_s$, find the eigenvalues of $(\varphi \otimes \psi)\colon V \otimes_k W \longrightarrow V \otimes_k W$.

(b) Even if φ and ψ aren't necessarily diagonalizable, prove that the same result is true. (Here the eigenvalues should be interpreted as the roots, with multiplicity, of the characteristic polynomial, although you probably won't be able to prove the result directly from this definition).

(c) Find a formula for $\mathrm{Tr}(\varphi \otimes \psi)$ in terms of $\mathrm{Tr}(\varphi)$ and $\mathrm{Tr}(\psi)$, and for $\det(\varphi \otimes \psi)$ in terms of $\det(\varphi)$ and $\det(\psi)$.

(SUGGESTION: For (a) and (b), choose good bases $\{e_i\}_{i=1}^{r}$ and $\{f_j\}_{j=1}^{s}$ for V and W respectively, and consider the matrix for $\varphi \otimes \psi$ written in terms of the basis $\{e_i \otimes f_j\}_{i=1,j=1}^{r,s}$ for $V \otimes W$.)