1. In this problem we will show that tensor product is a right-exact functor. Let $A$ be a ring, and suppose that we have an $A$-module $N$ and an exact sequence

$$
\begin{array}{cccccc}
M_1 & \xrightarrow{f} & M & \xrightarrow{g} & M_2 & \rightarrow & 0
\end{array}
$$

of $A$-modules. We need to show that

$$
\begin{array}{cccccc}
M_1 \otimes A N & \xrightarrow{\bar{f}} & M \otimes A N & \xrightarrow{\bar{g}} & M_2 \otimes A N & \rightarrow & 0
\end{array}
$$

is also exact, where $\bar{f} = f \otimes \text{Id}_N$, $\bar{g} = g \otimes \text{Id}_N$, and where the tensor product is over $A$.

(a) Start by showing that $\bar{g}$ is surjective. Since elements of the form $m_2 \otimes n$ generate $M_2 \otimes A N$, it is sufficient to show that these types of elements are in the image of $\bar{g}$. (Also: Don’t forget the working hypothesis in this question, that the original sequence is exact.)

(b) Now show that $\bar{g} \circ \bar{f}$ is the zero map. One way is again by seeing what happens to the generators $m_1 \otimes n$ of $M_1 \otimes A N$.

Let $Q = \text{Im}(\bar{f})$ so that $Q$ is a submodule of $M \otimes A N$ and set $P = (M \otimes A N)/Q$. By part (b), $Q \subseteq \ker(\bar{g})$ so that $\bar{g}$ factors through the projection map to $P$. That is, there exists $g_0: P \rightarrow M_2 \otimes A N$ such that the diagram below commutes.

$$
\begin{array}{cccccc}
M_1 \otimes A N & \xrightarrow{\bar{f}} & M \otimes A N & \xrightarrow{\bar{g}} & M_2 \otimes A N & \rightarrow & 0
\end{array}
$$

We want to show that the map $g_0$ is an isomorphism. To do this we look for a map $h: M_2 \otimes A N \rightarrow P$ in the other direction, and to do that we start as usual by looking for a bilinear map $M_2 \times A N \rightarrow P$.

(c) Given $m_2 \in M_2$ call a lift of $m_2$ any element $m$ of $M$ such that $g(m) = m_2$. Explain why the “map” $M_2 \times A N \rightarrow M \otimes A N$ given by $(m_2, n) \mapsto (\text{any lift of } m_2) \otimes n$ is not well-defined, i.e, is not a map.

(d) Now explain why the map $M_2 \times A N$ given by $(m_2, n) \mapsto \text{the class of } (\text{any lift of } m_2) \otimes n$ in $P$ is a well-defined map. (The map in this question is the ill-defined formula from (c), followed by projection to $P$.)
(e) Given that this map is well-defined (i.e., given part (d)) , to show that the map is A-bilinear.

Thus, the map above induces an A-module map $h: M_2 \otimes N \rightarrow P$ such that $h(m_2 \otimes n) =$ the image of $m \otimes n$ in $P$ (where $m$ is any lift of $m_2$). Using this formula show that

(f) $g_0 \circ h = \text{Id}_{M_2 \otimes N}$ and

(g) $h \circ g_0 = \text{Id}_P$.

2. We will now use right exactness to compute some tensor products.

(a) Let $m$ and $n$ be positive integers, and set $d = \gcd(m, n)$. Prove that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$.

Now fix a field $k$ and set $A = k[x, y]; M_1 = A/(x); M_2 = A/(x-y);$ and $M_3 = A/(x-1)$. All of $M_i$ are $A$-modules.

Compute

(b) $M_1 \otimes_A M_2$.

(c) $M_1 \otimes_A M_3$.

(d) $M_2 \otimes_A M_3$.

The modules in (a), (b), and (c) are finite dimensional vector spaces over $k$. When you compute them, try and describe them in the simplest way possible, and also give their dimensions as vector spaces over $k$.

3. In this problem we will examine some formulae from linear algebra from the point of view of multilinear algebra. Let $k$ be a field and $V$ and $W$ finite dimensional vector spaces over $k$. If we choose bases $\{v_j\}_{j=1}^{r}$, $\{w_i\}_{i=1}^{s}$ for $V$ and $W$, then any linear map $\varphi: V \rightarrow W$ can be represented (with respect to this basis) by a matrix $[c_{ij}]$. On the other hand, if we set $v_1^*, \ldots, v_r^*$ to be the basis dual to $v_1, \ldots, v_r$, then we know from class that $\{v_j^* \otimes w_i\}_{j=1, i=1}^{r, s}$ is a basis for $V^* \otimes W$. Hence any $\varphi \in \text{Hom}_k(V, W) \cong V^* \otimes W$ can be written as a unique sum $\sum_{i,j} c'_{ij}(v_j^* \otimes w_i)$.

(a) Show that $c_{ij} = c'_{ij}$ for all $i$ and $j$. I.e., show that the entries of the matrix give the coordinates of $\varphi$ in the basis $\{v_j^* \otimes w_i\}$. (SUGGESTION: Linear maps are determined by what they do to a basis. Compare what the matrix description and the sum $\sum_{i,j} c'_{ij}(v_j^* \otimes w_i)$ are telling us to do to each basis vector $v_j$.)
(b) Show that the evaluation map $W^* \times W \rightarrow k$ given by $(f, w) \mapsto f(w)$ is $k$-bilinear.

The evaluation map therefore induces a map $W^* \otimes W \rightarrow k$, and so (via the isomorphism $W^* \otimes W \cong \text{Hom}(W, W)$) we get a map $\text{Hom}(W, W) \rightarrow k$. This map does not depend on any choice of basis of $W$, since the original description of the evaluation map was basis free. Let us now try and figure out what this map is.

(c) Let $\varphi: W \rightarrow W$ be a linear map. Choosing a basis $w_1, \ldots, w_s$ for $W$, we can represent $\varphi$ as matrix $[c_{ij}]$ or (by part (a)) as the sum $\sum c_{ij} w_j^* \otimes w_i \in W^* \otimes W$. Use the second representation to evaluate the result of applying the map $W^* \otimes W \rightarrow k$ to $\varphi$. What (basis free) operation on matrices is this?

Let $U$, $V$ and $W$ be three (finite dimensional) vector spaces over $k$. By using the map $W^* \otimes W \rightarrow k$ as defined in the previous question, we get a map

\[
V^* \otimes_k (W \otimes_k W^*) \otimes_k U \rightarrow V \otimes_k k \otimes_k k \otimes_k U \cong V^* \otimes_k U.
\]

This gives us the following diagram, where the vertical maps are isomorphisms, and the bottom horizontal map the map just described.

\[
\begin{array}{ccc}
\text{Hom}(V, W) \otimes_k \text{Hom}(W, U) & \rightarrow & \text{Hom}(V, U) \\
\oplus \oplus & & \oplus \oplus \\
(V^* \otimes_k W) \otimes_k (W^* \otimes_k U) & \rightarrow & V^* \otimes_k U \\
\oplus \oplus & & \oplus \oplus \\
V^* \otimes_k (W \otimes_k W^*) \otimes_k U & \rightarrow & V^* \otimes_k U
\end{array}
\]

Thus, combining all these, we get a map $\text{Hom}(V, W) \otimes \text{Hom}(W, U) \rightarrow \text{Hom}(V, U)$. We will now work out what this map is. Choose bases $v_1, \ldots, v_r$ for $V$, $w_1, \ldots, w_s$ for $W$, and $u_1, \ldots, u_t$ for $U$.

(d) Given linear maps $\varphi: V \rightarrow W$ and $\psi: W \rightarrow U$, write out $\varphi$ and $\psi$ as elements of $V^* \otimes W$ and $W^* \otimes U$ respectively.

(e) Evaluate the image of $\varphi \otimes \psi$ in $V^* \otimes U$.

(f) What operation on matrices does this correspond to?