1. In this question we will complete the inductive step for the proof that, given a surjection of $A$-modules $\varphi: M \rightarrow N$, the kernel of $T^\bullet(\varphi): T^\bullet(M) \rightarrow T^\bullet(N)$ is the two-sided ideal of $T(M)$ generated by $P \subset T^1(M) = M$, with $P = \ker(\varphi)$. Recall that we have already shown that this statement is correct in degrees 1 and 2.

To demonstrate the general case, let $I_p$ be the kernel of $T(\varphi)_p: T^p(M) \rightarrow T^p(N)$. By induction we can assume that $I_p$ is the degree $p$ part of the two-sided ideal generated by $P$. We have the exact sequence

$$0 \rightarrow I_p \xrightarrow{i_p} T^p(M) \xrightarrow{T(\varphi)_p} T^p(N) \rightarrow 0.$$ 

Applying the functor $- \otimes M$ to the sequence above and the functor $T^p(N) \otimes -$ to the sequence

$$0 \rightarrow P \xrightarrow{i} M \xrightarrow{\varphi} N \rightarrow 0$$

we get two exact sequences which we can combine, as shown below.

$$T^{p+1}(M)$$

$\rightarrow$

$I_p \otimes M \xrightarrow{i_p \otimes \text{Id}_M} T^p(M) \otimes M \xrightarrow{T(\varphi)_p \otimes \text{Id}_M} T^p(N) \otimes M \rightarrow 0$

$T^p(N) \otimes P$

$\rightarrow$

$T^p(N) \otimes M \xrightarrow{\text{Id}_{T^p(N)} \otimes 1} T^p(N) \otimes N \rightarrow 0$

$= T^{p+1}(N)$

Complete the argument: Show that the kernel of $T^\bullet(\varphi)$ in degree $p + 1$ is the degree $p + 1$ piece of the two sided ideal generated by $P$.

2. Let $A = \mathbb{Z}$, $n$ a positive integer, and $N = \mathbb{Z}/n\mathbb{Z}$. Use the theorem from question 1 to compute $T^\bullet(N)$. In particular, explicitly describe the $\mathbb{Z}$-module $T^p(N)$ for each $p \geq 1$, and describe the multiplication map $T^p(N) \otimes T^q(N) \rightarrow T^{p+q}(N)$. 


3. We know that the functor $T^*$ takes a surjection to a surjection; in this question we see that $T^*$ does not in general take injections to injections.

(a) Let $A = \mathbb{Z}$, $N = \mathbb{Z}/4\mathbb{Z}$, $M$ the submodule of $N$ generated by $2 \in N$, and $\varphi : M \hookrightarrow N$ the inclusion map. Show that $T^*(\varphi) : T^*(M) \rightarrow T^*(N)$ is not an injection.

Let us return to the case that $A$ is a general commutative ring. An $A$-submodule $M$ of $N$ is called a direct factor if there exists $j : N \rightarrow M$ such that $j \circ i = \text{Id}_M$, where $i : M \hookrightarrow N$ is the inclusion map.

(b) Show that if $M$ is a direct factor of $N$ then $T^*(i) : T^*(M) \rightarrow T^*(N)$ is injective. (Suggestion: $T^*$ is a functor!)

Remark: This homework assignment should be substantially easier than the others. The only questions I could think of about the tensor algebra which were more difficult than the ones on this assignment seemed too difficult. I couldn’t think of any intermediate questions!