1. In this question we will complete the inductive step for the proof that, given a surjection of \( A \)-modules \( \varphi: M \to N \), the kernel of \( T^\bullet(\varphi): T^\bullet(M) \to T^\bullet(N) \) is the two-sided ideal of \( T(M) \) generated by \( P \subset T^1(M) = M \), with \( P = \ker(\varphi) \). Recall that we have already shown that this statement is correct in degrees 1 and 2.

To demonstrate the general case, let \( I_p \) be the kernel of \( T(\varphi)_p: T^p(M) \to T^p(N) \). By induction we can assume that \( I_p \) is the degree \( p \) part of the two-sided ideal generated by \( P \). We have the exact sequence

\[
0 \to I_p \xrightarrow{i_p} T^p(M) \xrightarrow{T(\varphi)_p} T^p(N) \to 0.
\]

Applying the functor \(- \otimes M\) to the sequence above and the functor \( T^p(N) \otimes -\) to the sequence

\[
0 \to P \xrightarrow{i} M \xrightarrow{\varphi} N \to 0
\]

we get two exact sequences which we can combine, as shown below.

```
\begin{tikzpicture}
  \node (A) at (0,0) {\( T^{p+1}(M) \)};
  \node (B) at (3,0) {\( T^p(N) \otimes P \)};
  \node (C) at (3,-3) {\( T^p(N) \otimes M \)};
  \node (D) at (6,-3) {\( T^p(N) \otimes N \)};
  \node (E) at (6,-6) {\( 0 \)};

  \draw[->] (A) -- node[above] {\( i_p \otimes \text{Id}_M \)} (B);
  \draw[->] (B) -- node[above] {\( \text{Id}_{T^p(N)} \otimes 1 \)} (C);
  \draw[->] (C) -- node[above] {\( T^p(M) \otimes M \)} (D);
  \draw[->] (D) -- node[above] {\( 0 \)} (E);

  \draw[->] (A) -- node[left] {\( \text{Id}_{T^p(M)} \otimes \text{Id} \)} (C);
  \draw[->] (B) -- node[left] {\( T(\varphi)_p \otimes \text{Id}_M \)} (D);

\end{tikzpicture}
```

Complete the argument: Show that the kernel of \( T^\bullet(\varphi) \) in degree \( p + 1 \) is the degree \( p + 1 \) piece of the two sided ideal generated by \( P \).

2. Let \( A = \mathbb{Z} \), \( n \) a positive integer, and \( N = \mathbb{Z}/n\mathbb{Z} \). Use the theorem from question 1 to compute \( T^\bullet(N) \). In particular, explicitly describe the \( \mathbb{Z} \)-module \( T^p(N) \) for each \( p \geq 1 \), and describe the multiplication map \( T^p(N) \otimes T^q(N) \to T^{p+q}(N) \).
3. We know that the functor $T^*$ takes a surjection to a surjection; in this question we see that $T^*$ does not in general take injections to injections.

(a) Let $A = \mathbb{Z}$, $N = \mathbb{Z}/4\mathbb{Z}$, $M$ the submodule of $N$ generated by $\overline{2} \in N$, and $\varphi : M \rightarrow N$ the inclusion map. Show that $T^*(\varphi) : T^*(M) \rightarrow T^*(N)$ is not an injection.

Let us return to the case that $A$ is a general commutative ring. An $A$-submodule $M$ of $N$ is called a direct factor if there exists $j : N \rightarrow M$ such that $j \circ i = \text{Id}_M$, where $i : M \rightarrow N$ is the inclusion map.

(b) Show that if $M$ is a direct factor of $N$ then $T^*(i) : T^*(M) \rightarrow T^*(N)$ is injective. (SUGGESTION: $T^*$ is a functor!)

Remark: This homework assignment should be substantially easier than the others. The only questions I could think of about the tensor algebra which were more difficult than the ones on this assignment seemed too difficult. I couldn’t think of any intermediate questions!