Throughout the assignment we work over an algebraically closed field \( k \) of characteristic zero.

Character is destiny.  
— Heraclitus

1. Let \( G \) be a finite group with irreducible representations \( W_1, \ldots, W_h \), and let \( V \) be a finite dimensional representation of \( G \). As usual we can decompose \( V \) as a direct sum of irreducibles:

\[
V = m_1 W_1 \oplus m_2 W_2 \oplus \cdots \oplus m_h W_h.
\]

This decomposition is not usually unique. For instance, imagine that \( V \) is three-dimensional, so that the trivial representation “appears three times”. In decomposing \( V \) as a direct sum of irreducibles we must make a choice how to decompose \( V \) into three one-dimensional subspaces, and there are many ways to do that.

However, there an aspect of uniqueness to the decomposition. For any irreducible \( W_i \), the subspace spanned by all the components isomorphic to \( W_i \) is uniquely determined (e.g., the subspace \( V^G \), the sum of all the trivial representations, is uniquely determined). These components are called the isotypic components. In this problem we will prove that the isotypic components are unique by finding the \( G \)-homomorphism projectors onto them.

Let \( d_1, \ldots, d_h \) be the dimensions of the irreducible representations, and \( \chi_1, \ldots, \chi_h \) their characters. For each \( i, i = 1, \ldots, h \), set \( f_i = d_i \chi_i \). Also recall the construction of the \( G \)-endomorphisms \( \varphi_f \) for any class function \( f \).

(a) For each \( i \), show that \( \varphi_f \) acts as the identity on \( W_i \), and as the zero map on \( W_j \) when \( j \neq i \).

(b) For any representation \( V \) show that the \( G \)-endomorphism \( \varphi_f \) is a projection map with image the subspace of \( V \) spanned by the irreducibles of type \( W_i \).

Remark. When \( W_i \) is the trivial representation, \( d_i = 1 \) and \( \chi_i = 1 \), so that \( f_i = 1 \) and \( \chi_f = \chi_1 = \text{Avg}_G \) is the \( G \)-averaging operator. I.e., the operators above generalize the averaging operator which, as we have seen, is projection onto the fixed subspace \( V^G \).

2. Let \( G \) be a finite group, \( X \) a finite set with \( G \)-action, \( (V, \rho) \) the corresponding permutation representation of \( G \), and \( \chi \) its character.

(a) Show that the average number of fixed points an element \( g \in G \) has on \( X \) is equal to the number of times that \( V \) contains the trivial representation. (Suggestion: write down a formula for the average number of fixed points and reinterpret this as a pairing \( \langle \cdot, \cdot \rangle \) between characters.)
(b) Show that the number of times that $V$ contains the trivial representation is also equal to the number of orbits of $G$ on $X$. (**Suggestion:** Find a bijection between orbits and a set of basis vectors of $V^G$.)

**Remark.** (a) and (b) together show that the average number of fixed points of $G$ acting on $X$ is equal to the number of orbits, i.e., this gives a representation-theory proof of a result known as “Burnside’s lemma”.

(c) Suppose that $G = \Sigma_n$ the symmetric group, $X = \{1, 2, \ldots, n\}$, that $G$ acts on $X$ by the usual permutations, and let $V$ be the associated permutation representation. We have seen that $V$ splits as a direct sum of the trivial representation and an irreducible $(n-1)$-dimensional representation, called the **standard representation**. Prove that $\chi_{\text{std}}(g) = \#\{\text{fixed points of } g\} - 1$ for all $g \in G$.

3. Let $V_{\text{std}}$ be the standard representation of $\Sigma_n$ as defined H7 Q1 (or Q2(c) above). In this problem we will show that $\Lambda^k V_{\text{std}}$ is irreducible for $k = 1, \ldots, n-1$.

(a) Suppose that for a representation $W$ we know that $\langle \chi_W, \chi_W \rangle = 2$. Show that $W$ is the direct sum of two irreducible representations, each appearing with multiplicity one.

(b) Suppose that $V$ is a vector space and that $V = V_1 \oplus V_2$ where $V_1$ is a one-dimensional vector space. Show that $\Lambda^k V = (V_1 \otimes \Lambda^{k-1} V_2) \oplus \Lambda^k V_2$. (Don’t forget results we already know from class.)

(c) Let $V$ be the permutation representation as in 2(c) and compute $\langle \chi_{\Lambda^k V}, \chi_{\Lambda^k V} \rangle$.

(d) Use the results above to show that $\Lambda^k V_{\text{std}}$ is irreducible.

Part (c) will require some non-trivial combinatorics.